Lecture 15: Quadratic Forms

Example:

Classify the quadratic function

$$Q(x,y) = 5x^2 - 4xy + 5y^2$$

Idea:

Complete the square:

$$Q(x,y) = 5\left(x^2 - \frac{4}{5}xy + y^2\right)$$

$$= 5\left(x^2 - \frac{4}{5}xy + \frac{4}{25}y^2 - \frac{4}{25}y^2 + y^2\right)$$

$$= 5\left(\left(x - \frac{2}{5}y\right)^2 + \frac{21y^2}{25}\right)$$

$$= 5\left(x - \frac{2}{5}y\right)^2 + \frac{21y^2}{5} \Rightarrow \text{ellipse.}$$

Minium exists and occurs when $x=0, y=0$

How do we sketch $Q(x,y) = 1$?

Another Idea:

If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ then

$$[x\ y]\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x\ y]\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = ax^2 + bxy + cxy + dy^2$$

$\Rightarrow$ All quadratic functions can be mapped to a matrix

$$Q(x,y) = \langle \vec{x}, A\vec{x} \rangle = \vec{x}^* A \vec{x}$$

For our example

$$A = \begin{bmatrix} 5 & -4 \\ 0 & 5 \end{bmatrix} \text{ or } A = \begin{bmatrix} 5 & 0 \\ -4 & 5 \end{bmatrix} \text{ or } A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

The last one is Hermition and is thus easier to analyze.

If A is Hermitian then

$$A = U \Lambda U^*$$

Let λ_i be the i-th eigenvalue of A and $\vec{y} = U^* \vec{x}$.

$$\Downarrow$$

$$\begin{aligned} Q(\vec{x}) &= \vec{x}^* A \vec{x} \\ &= \vec{x}^* U \Lambda U^* \vec{x} \\ &= \vec{y}^* \Lambda \vec{y}^* \\ &= \underline{\lambda_1 |y_1|^2 + \dots + \lambda_n |y_n|^2} \end{aligned}$$

Completing the square.

Returning to the example:

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 5 \end{bmatrix}\right)$$

$$= (\lambda - 5)^2 - 4$$

$$\Rightarrow \lambda_i = 3, 7$$

$\underline{\lambda_1 = 3}$

$$\lambda I - A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\underline{\lambda_2 = 5}$

$$\lambda I - A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U \Lambda U^*$$

$$\Rightarrow Q(\vec{x}) = \vec{x}^* A \vec{x}$$

$$= \vec{x}^* U \Lambda U^* \vec{x}$$

Let $\vec{y} = U^* \vec{x}$ we have that

$$Q(\vec{x}) = \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2$$

$$= \frac{3}{\sqrt{2}} (x - y)^2 + \frac{5}{\sqrt{2}} (x + y)^2$$

If we sketch

$$Q(\vec{x}) = \sqrt{2}$$

we have that

$$3(x-y)^2 + 5(x+y)^2 = 2$$

Letting $x=y$ we have

$$20x^2 = 2$$

$$\Rightarrow x = 1/\sqrt{10}$$

Letting $x=-y$ we have

$$12x^2 = 2$$

$$x = 1/\sqrt{6}$$