Lecture 16: Gershgorin Circle Theorem

Definition – Let $A \in M_{n \times n}(\mathbb{C})$. For each $1 \leq i \leq n$, define the i-th Gershgorin disk:

$$D_i = \{ z \in \mathbb{C} : |z - a_{ii}| \leq r_i \}, \quad r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

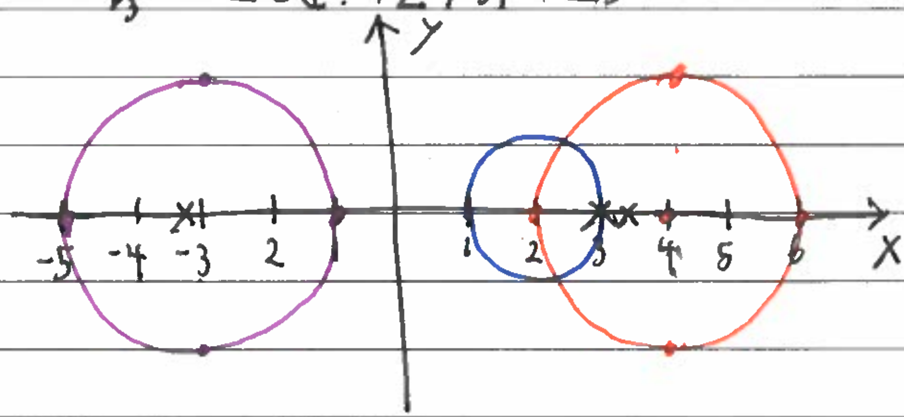
The Gershgorin domain $D_A = \bigcup_{i=1}^{n} D_i$ is the union of the Gershgorin disks.

Theorem – If λ is an eigenvalue of A then $\lambda \in D_A$.

Example:

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & -1 \\ -1 & -1 & -3 \end{bmatrix}$

$$\Rightarrow D_1 = \{ z \in \mathbb{C} : |z-2| \leq 1 \}$$
$$D_2 = \{ z \in \mathbb{C} : |z-4| \leq 2 \}$$
$$D_3 = \{ z \in \mathbb{C} : |z+3| \leq 2 \}$$



$$\lambda_1 = 3, \quad \lambda_2 = \sqrt{10}, \quad \lambda_3 = -\sqrt{10}$$

proof of theorem:

Let $\vec{v}$ be an eigenvector of A with components

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Let v_i satisfy $|v_i| \geq |v_j|$ for all $j \in \{1, \ldots, n\}$.

Therefore,

$$A\vec{v} = \lambda \vec{v}$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} v_j = \lambda v_i$$

$$\Rightarrow \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij} v_j = (\lambda - a_{ii}) v_i$$

$$\Rightarrow |(\lambda - a_{ii})| \cdot |v_i| = \left| \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij} v_j \right| \leq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \cdot |v_j|$$

$$\Rightarrow |\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \cdot \frac{|v_j|}{|v_i|} \leq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|.$$

Definition - A square matrix A is called strictly diagonally dominant if

$$|a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

Theorem - A strictly diagonally dominant matrix is nonsingular, i.e., invertible.

Example:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 4 & 2 \\ -2 & -1 & 5 \end{bmatrix}$$

$$D_1 = \{ z \in \mathbb{C} : |z-3| \leq 2 \}$$

$$D_2 = \{ z \in \mathbb{C} : |z-4| \leq 3 \}$$

$$D_3 = \{ z \in \mathbb{C} : |z-5| \leq 3 \}$$

$\Rightarrow A$ is invertible.