Lecture 2: Vector Spaces and Subspaces

Definition - A field F is a set with operations $+$ and $\cdot$ such that the following hold

1. Closure: For all $a, b \in F$
   $$a+b \in F, \quad a \cdot b \in F$$
2. Commutativity: For all $a, b \in F$
   $$a+b=b+a, \quad a \cdot b = b \cdot a.$$
3. Associativity: For all $a, b, c \in F$
   $$(a+b)+c=a+(b+c), \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$
4. Additive and Multiplicative Identity: There exists $0, 1 \in F$ such that for all $a \in F$
   $$0+a=a \quad \text{and} \quad 1 \cdot a = a.$$
5. Additive inverse: For all $a \in F$ there exists $-a \in F$ such that $a+(-a)=0$.
6. Multiplicative inverse: For every $a \neq 0$ in F, there exists $a^{-1} \in F$ such that $a^{-1}a=1$.
7. Distributivity: For all $a, b, c \in F$
   $$a \cdot (b+c) = a \cdot b + a \cdot c$$

Example (Standard Operations)

1. $[-1, 1]$, not a field
2. $\mathbb{Z}$, not a field
3. $\mathbb{Q}$, is a field
4. $\mathbb{R}$, is a field
5. $\mathbb{C}$, is a field
6. The set of all polynomials is not a field.
7. The set of rational functions is a field.

Definition - A set V is a vector space over a field F if there are operations of addition and scalar multiplication on V such that the following hold.

For all $\vec{u}, \vec{v}, \vec{w} \in V$, $a, b \in F$

1. $\vec{u}+\vec{v} \in V$
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. There exists $\vec{0} \in V$, so that $\vec{0}+\vec{v}=\vec{v}$.
5. There is $(-\vec{u}) \in V$ such that $\vec{u}+(-\vec{u})=\vec{0}$
6. $a\vec{u} \in V$
7. $a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$
8. $(a+b)\vec{u}=a\vec{u}+b\vec{u}$
9. $a(b\vec{u})=ab\vec{u}$
10. $1\vec{u}=\vec{u}$

Examples:

1. $\mathbb{R}^n$ = vectors of real numbers;

$$\vec{v}=\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

is a vector space over $\mathbb{R}$

2. $M_{m\times n}(\mathbb{R})$ = $m\times n$ matrices of real numbers is a vector space over $\mathbb{R}$.
3. $P_n(\mathbb{R})$ polynomials of degree $\leq n$ with coefficients in F.
4. $C(\mathbb{R})$ continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition - A subset W of V is a subspace of V if W is also a vector space under scalar multiplication and addition in V.

Theorem - Let W be a subset of V. W is a subspace of V if and only if:

(i) $\vec{0} \in W$

(ii) $\vec{u}, \vec{v} \in W \Rightarrow \vec{u}+\vec{v} \in W$

(iii) $\vec{u} \in W \Rightarrow a\vec{u} \in W$, for $a \in F$

Example

$W = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x \}$. W is a subspace of $\mathbb{R}^2$

proof:

(i) Clearly, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$

(ii) Let $\vec{v}_1, \vec{v}_2 \in W$. Therefore, there exists $v_1, v_2 \in \mathbb{R}$ such that

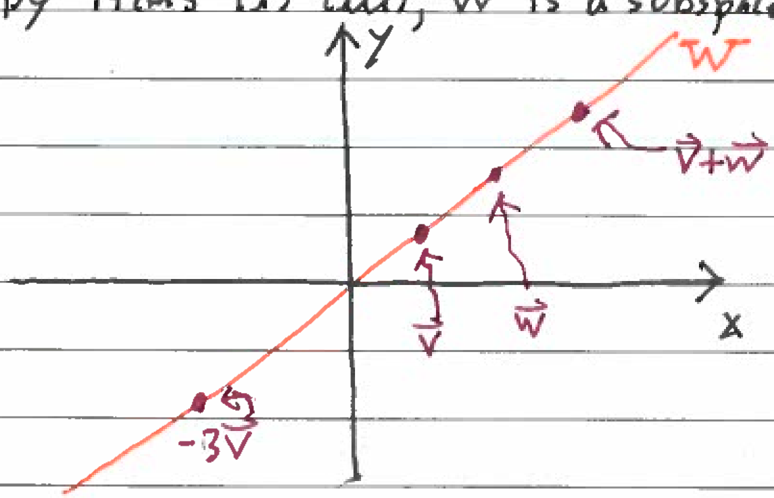
$$\vec{v}_1 = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1+v_2 \\ v_1+v_2 \end{bmatrix} \in W$$

(iii) Let $\vec{v} \in W$ and $a \in \mathbb{R}$. Therefore, there exists v such that $\vec{v} = \begin{bmatrix} v \\ v \end{bmatrix}$. Consequently,

$$a\vec{v} = a\begin{bmatrix} v \\ v \end{bmatrix} = \begin{bmatrix} av \\ av \end{bmatrix} \in W.$$

By items (i)-(iii), W is a subspace of $\mathbb{R}^2$.



*Subspaces can be thought of as lines and/or planes through the origin.

Example:

1. $W = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0, y \geq 0 \}$ is not a subspace of $\mathbb{R}^2$.

proof

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W \Rightarrow -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin W$.

2. $W = \{2\times 2$ symmetric matrices$\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

proof:

(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a symmetric matrix

(ii) Let $A, B \in W$. Therefore, there exists $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ such that

$$A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix}$$

$$\Rightarrow A + B = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_2+b_2 & a_3+b_3 \end{bmatrix} \in W.$$

(iii) Let $A \in W$ and $\lambda \in \mathbb{R}$. Therefore, there exists $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

$$\Rightarrow \lambda A = \begin{bmatrix} \lambda a_1 & \lambda a_2 \\ \lambda a_2 & \lambda a_3 \end{bmatrix} \in W.$$

By items (i)-(iii) W is a subspace.