Lecture #4: Linear Transformations

<u>Definition</u>- If $T: V \to W$ is a function from a vector space V into a vector space W, then T is called a linear transformation from V to W if for all $\vec{u}, \vec{v} \in V$ and $c \in F$

(i) $T[\vec{u}+\vec{v}] = T[\vec{u}] + T[\vec{v}]$

(ii) $T[c\vec{u}] = cT[\vec{u}]$.

<u>Example:</u>

$T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T[c_0 + c_1 x + c_2 x^2] = c_0 + c_1(3x-5) + c_2(3x-5)^2$$

is linear.

proof:

Let $\vec{u}, \vec{v} \in P_2(\mathbb{R})$ then there exists $c_0, c_1, c_2, d_0, d_1, d_2$ such that $\vec{u} = c_0 + c_1 x + c_2 x^2$, $\vec{v} = d_0 + d_1 x + d_2 x^2$. Consequently,

(i)
$$\begin{aligned}
T[\vec{u}+\vec{v}] &= T[c_0 + c_1 x + c_2 x^2 + d_0 + d_1 x + d_2 x^2] \\
&= T[(c_0+d_0) + (c_1+d_1)x + (c_2+d_2)x^2] \\
&= (c_0+d_0) + (c_1+d_1)(3x-5) + (c_2+d_2)(3x-5)^2 \\
&= c_0 + c_1(3x-5) + c_2(3x-5)^2 + d_0 + d_1(3x-5) + d_2(3x-5)^2 \\
&= T[c_0 + c_1 x + c_2 x^2] + T[d_0 + d_1 x + d_2 x^2] \\
&= T[\vec{u}] + T[\vec{v}]
\end{aligned}$$

(ii)
$$\begin{aligned}
T[\lambda u] &= T[\lambda(c_0 + c_1 x + c_2 x^2)] \\
&= T[\lambda c_0 + \lambda c_1 x + \lambda c_2 x^2] \\
&= \lambda c_0 + \lambda c_1(3x-5) + \lambda c_2(3x-5)^2 \\
&= \lambda(c_0 + c_1(3x-5) + c_2(3x-5)^2) \\
&= \lambda T[c_0 + c_1 x + c_2 x^2] \\
&= \lambda T[\vec{u}].
\end{aligned}$$

By items (i)-(ii), T is a linear transformation.

Example:

The set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

is a basis for $\mathbb{R}^3$. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T[\vec{v}_1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T[\vec{v}_2] = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad T[\vec{v}_3] = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Find a formula for $T\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; then use this formula to compute:

$$T\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}.$$

We first express $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + c_2 + c_3 &= x \\ c_1 + c_2 &= y \\ c_1 &= z \end{aligned} \quad \Rightarrow c_1 = z, \quad c_2 = y - z, \quad c_3 = x - y$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (y-z)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x-y)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = zT\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (y-z)T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x-y)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= z\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (y-z)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (x-y)\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4x - 2y - z \\ 3x - 4y + z \end{bmatrix}.$$

$$\Rightarrow T\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \end{bmatrix}, \quad \text{Equivalently,} \quad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example:

Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by

$$T[c_0 + c_1 x + c_2 x^2] = c_0 + c_1(3x-5) + c_2(3x-5)^2$$

A basis for P_2 is $\{1, x, x^2\} = \beta$. We can make a correspondence with $\mathbb{R}^3$:

$$1 \cong \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad x \cong \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad x^2 \cong \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T[1] = 1 \cong \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T[x] = 3x-5 \cong \begin{bmatrix}-5\\3\\0\end{bmatrix}$$

$$T[x^2] = (3x-5)^2 \cong \begin{bmatrix}25\\-30\\9\end{bmatrix}$$

$$\Rightarrow T[c_0 + c_1 x + c_2 x^2] \cong c_0 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_1 \begin{bmatrix}-5\\3\\0\end{bmatrix} + c_2 \begin{bmatrix}25\\-30\\9\end{bmatrix}$$

$$= \begin{bmatrix} c_0 - 5c_1 + 25c_2 \\ 3c_1 - 30c_2 \\ 9c_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}}_{\text{matrix representation of operator}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

* For every linear transformation, what matters is what is the image of the basis vectors.

☆ In the matrix representation, the columns are the images of the basis vectors. ☆

Example:

$T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T[a_0 + a_1 x + a_2 x^2] = \begin{bmatrix} a_0 + a_1 & a_0 - a_1 \\ a_1 & 2a_2 \end{bmatrix}$$

The standard basis for $P_2(\mathbb{R})$:

$$\beta_1 = \{1, x, x^2\}$$

The standard basis for $M_{2\times 2}(\mathbb{R})$:

$$\beta_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow T[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T[x] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cong \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T[x^2] = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow T[a_0 + a_1 x + a_2 x^2] \cong \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$