# Lecture 5: Nullity, Rank, and Isomorphisms

**Definition:** Let $T: V \to W$ be a linear transformation. The nullspace or kernel of $T$ is defined by

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = 0\}.$$

The image of $T$ is defined by:

$$\mathrm{im}(T) = \{\vec{w} \in W : T(\vec{v}) = \vec{w} \text{ for some } \vec{w}\}.$$

The nullity of $T$ is the dimension of $\ker(T)$ while the rank of $T$ is the dimension of $\mathrm{im}(T)$.

**Example:**

Let $T: P_2(\mathbb{R}) \to \mathbb{R}^2$ be defined by

$$T[p] = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

Let $p(t) = a_0 + a_1 t + a_2 t^2$. Therefore,

$$T[p] = T[a_0 + a_1 t + a_2 t^2]$$
$$= \begin{bmatrix} a_0 \\ a_0 + a_1 \end{bmatrix}$$

- $T[p] = 0$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_0 + a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a_0 = 0 \\ a_0 + a_1 = 0 \end{matrix} \Rightarrow \begin{matrix} a_0 = 0 \\ a_1 = 0 \end{matrix}, \quad a_2 = \text{anything}$$

Therefore,

$$\ker(T) = \{p \in P_2(\mathbb{R}) : p(t) = a_2 t^2\}$$
$$= \mathrm{span}\{t^2\}$$

$$\Rightarrow \text{nullity}(T) = 1$$

- Since

$$T[a_0 + a_1 t + a_2 t^2] = \begin{bmatrix} a_0 \\ a_0 + a_1 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

it follows that

$$\mathrm{im}(T) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$\Rightarrow \text{rank}(T) = 2.$$

Example:

$T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = (a_{11}+a_{12}+a_{21}+a_{22}) + (a_{11}+a_{12}+a_{21}+a_{22})t + 0t^2$$

- $T(\vec{v}) = 0$

$$\Rightarrow a_{11}+a_{12}+a_{21}+a_{22} = 0$$
$$a_{11}+a_{12}+a_{21}+a_{22} = 0$$
$$\Rightarrow a_{11} = -a_{12}-a_{21}-a_{22}$$
$$\Rightarrow \begin{bmatrix} -a_{12}-a_{21}-a_{22} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{12}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\Rightarrow \ker(T) = \operatorname{span}\left\{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$
$$\Rightarrow \text{nullity} = 3$$

- $T(\vec{v}) = (a_{11}+a_{12}+a_{21}+a_{22}) + (a_{11}+a_{12}+a_{21}+a_{22})t + 0t^2$

$$= a_{11}(1+t) + a_{12}(1+t) + a_{21}(1+t) + a_{22}(1+t)$$

$$\Rightarrow \operatorname{im}(T) = \operatorname{span}\{1+t, 1+t, 1+t, 1+t\}$$
$$= \operatorname{span}\{1+t\}$$

$$\Rightarrow \text{rank} = 1.$$

Theorem - Let $T: V \to W$ be a linear transformation. Then,

$$\dim(V) = \text{nullity}(T) + \text{rank}(T).$$

~~proof~~:

Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for $\ker(T)$. Expand $\beta$ to a basis $\{\vec{v}_1, \ldots, \vec{v}_n, \vec{w}_1, \ldots, \vec{w}_m\}$ for $V$. Therefore, for all $\vec{v} \in V$, there exists $a_1, \ldots, a_n \in F$, $b_1, \ldots, b_m \in F$ such that

$$\vec{v} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n + b_1\vec{w}_1 + \ldots + b_m\vec{w}_m$$

$$\Rightarrow T[\vec{v}] = b_1 T[\vec{w}_1] + \ldots + b_m T[\vec{w}_m].$$

$$\Rightarrow \text{im}(T) = \text{span}\{T[\vec{w}_1], \ldots, T[\vec{w}_m]\}.$$

Now, suppose there exists constants $c_1, \ldots, c_m \in F$ such that

$$c_1 T[\vec{w}_1] + \ldots + c_m T[\vec{w}_m] = 0$$

$$\Rightarrow T[c_1\vec{w}_1 + \ldots + c_m\vec{w}_m] = 0$$

$$\Rightarrow c_1\vec{w}_1 + \ldots + c_m\vec{w}_m \in \ker(T)$$

However, $c_1\vec{w}_1 + \ldots + c_m\vec{w}_m \in \text{span}\{\vec{w}_1, \ldots, \vec{w}_m\}$ and

$$\ker(T) \cap \text{span}\{\vec{w}_1, \ldots, \vec{w}_m\} = 0$$

$$\Rightarrow c_1\vec{w}_1 + \ldots + c_m\vec{w}_m = 0$$

$$\Rightarrow c_1 = \ldots = c_m = 0$$

Since $\vec{w}_1, \ldots, \vec{w}_m$ are linearly ind. ∎

Definition - A function $T: V \to W$ is called an isomorphism if $T$ is one-to-one and onto $W$. We say $V$ and $W$ are isomorphic, denoted $V \cong W$.

Theorem - A linear transformation is one-to-one if and only if $\ker(T) = 0$

proof:

(i) Suppose $T$ is one-to-one and let $\vec{v} \in V$ satisfy $T[v] = 0$. Since $T[0] = 0$ it follows that $0 = \vec{v}$.

(ii) Suppose $\ker(T) = 0$ and there exists $\vec{v}_1, \vec{v}_2 \in V$ such that $T[\vec{v}_1] = T[\vec{v}_2]$. Therefore, $T[\vec{v}_1 - \vec{v}_2] = 0 \Rightarrow \vec{v}_1 - \vec{v}_2 = 0 \Rightarrow \vec{v}_1 = \vec{v}_2$. ∎

Corollary - Let $T: V \to W$ be a linear transformation with $\dim(V) = \dim(W)$. Then $T$ is one-to-one if and only if $T$ maps onto $W$.

proof:

$T$ is one-to-one if and only if $\ker(T) = 0$. Therefore,

$$\text{nullity}(T) = 0 \Leftrightarrow \text{rank} = \dim(V) = \dim(W).$$

Theorem - $V \cong W$ if and only if $\dim(V) = \dim(W)$.

Example: $M_{2\times2}(\mathbb{R}) \cong \mathbb{R}^4$.

$$M_{2\times2}(\mathbb{R}) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathbb{R}^4 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Define $I: M_{2\times2}(\mathbb{R}) \to \mathbb{R}^4$ by

$$I\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, I\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, I\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, I\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$I$ is an isometry that maps between the standard basis vectors.