Lecture #8: Conditions for Diagonalization

Theorem - Let T be a linear transformation on a vector space V. If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of T, and if $v_1, \dots, v_k$ are the corresponding eigenvectors of T then $\{v_1, \dots, v_k\}$ is a set of linearly independent vectors.

proof:

Let $P(m)$ be the logical statement that $\{\vec{v}_1, \dots, \vec{v}_m\}$ are linearly independent.

1. $P(1)$ is the statement $\{\vec{v}_1\}$ is linearly independent which is clearly true.
2. Suppose $P(m)$ is true. Therefore, $\{\vec{v}_1, \dots, \vec{v}_m\}$ are linearly independent. Now consider the equation

$$c_1\vec{v}_1 + \dots + c_{m+1}\vec{v}_{m+1} = 0 \quad (1)$$

$$\Rightarrow c_1\lambda_1\vec{v}_1 + \dots + c_{m+1}\lambda_{m+1}\vec{v}_{m+1} = 0 \quad (2)$$

Multiplying the first equation by λ_{m+1} we have

$$c_1\lambda_{m+1}\vec{v}_1 + \dots + c_{m+1}\lambda_{m+1}\vec{v}_{m+1} = 0$$

$$\Rightarrow c_1(\lambda_1 - \lambda_{m+1})\vec{v}_1 + \dots + c_m(\lambda_m - \lambda_{m+1})\vec{v}_m = 0$$

Since $\{\vec{v}_1, \dots, \vec{v}_m\}$ are linearly independent and the eigenvalues are distinct we have that $c_1 = \dots = c_m = 0$. Therefore, by (1) we have that $c_{m+1} = 0$ as well.

By items 1-2 and the principle of mathematical induction it follows that $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent.

Corollary - If V has dimension n, and if T has n distinct eigenvalues then T is diagonalizable.

Example

Let $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, Therefore,

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -2 \\ -2 & \lambda \end{pmatrix} = \lambda^2 - 4 \Rightarrow \lambda = \pm 2.$$

$\lambda_1 = 2$

$$\lambda I - A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \quad \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = -2$

$$\lambda I - A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, Therefore

$$[I(\mathcal{C}, S)] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = V.$$

$$[I(S, \mathcal{C})] = [I(\mathcal{C}, S)]^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$[T(\mathcal{C}, \mathcal{C})] = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\Rightarrow [T(\mathcal{C}, \mathcal{C})] = [I(S, \mathcal{C})][T(S, S)][I(\mathcal{C}, S)]$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Another way to look at this:

$\rightarrow$ Let $V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\Rightarrow AV = \Lambda V$ (eigenvalue equation in matrix form)

$\quad = V\Lambda$ (diagonal matrices commute)

$\Rightarrow A = V \Lambda V^{-1}$

$\Rightarrow \Lambda = V^{-1} A V.$

Example:

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The eigenvalue is $\lambda = 1$.

$\Rightarrow \lambda I - A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$. The only eigenvector is $\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Consequently, A is not diagonalizable.

Definition - Let λ be an eigenvalue of T. The algebraic multiplicity $m(\lambda)$ is the number of times λ is a root of the characteristic polynomial.

Theorem - Let $T: V \to V$ be a linear transformation with eigenvalues $\{\lambda_1, \dots, \lambda_k\}$. Then T is diagonalizable if and only if $\dim(E_{\lambda_i}) = m(\lambda_i)$ for all $i \in \{1, \dots, k\}$.