Lecture 9: Complex Inner Product Spaces

Complex Numbers

1. We say $z \in \mathbb{C}$ if there exists $a, b \in \mathbb{R}$ such that
$$z = a + ib$$
where $i^2 = -1$.
2. $a = Re(z)$, (Real Part)
3. $b = Im(z)$, (Imaginary Part)
4. $\bar{z} = z^* = a - ib$ (Complex Conjugate)
5. $z^* z = (a+ib)(a-ib) = a^2 + iab - iab - (i)^2 b = a^2 + b^2$
$\Rightarrow z^* z = |z|^2 = a^2 + b^2$
$\Rightarrow |z| = \sqrt{a^2 + b^2}$ (modulus)

Properties

1. $\overline{zw} = \bar{z} \cdot \bar{w}$

~~proof~~:

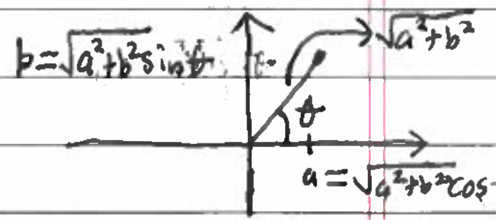
$z = a + ib$, $w = c + id$

$\Rightarrow \overline{z \cdot w} = \overline{(a+ib)(c+id)}$
$= \overline{(ac + ibc + ida - bd)}$
$= (ac - bd) - i(bc + da)$

$\Rightarrow \bar{z} \cdot \bar{w} = \overline{a+ib} \cdot \overline{c+id}$
$= (a - ib)(c - id)$
$= ac - iad - ibc - bd$
$= (ac - bd) - i(ad + bc)$.

2. $\overline{z+w} = \bar{z} + \bar{w}$.

Euler's Formula



$$a+bi=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}+i\frac{b}{\sqrt{a^2+b^2}}\right)$$
$$=\sqrt{a^2+b^2}\cos\theta+i\sqrt{a^2+b^2}\sin\theta,$$

where $\tan\theta=b/a$. Letting $r=\sqrt{a^2+b^2}$ we have:

$$a+bi=r\cos\theta+ir\sin\theta$$
$$=r(1-\theta^2/2+\theta^4/4!+\dots)+ir(\theta-\theta^3/3!+\theta^5/5!+\dots)$$
$$=r(1+(i\theta)^2/2+(i\theta)^4/4!+\dots)+r(i\theta+(i\theta)^3/3!+(i\theta)^5/5!+\dots)$$
$$=r(1+i\theta+(i\theta)^2/2+(i\theta)^3/3!+\dots)$$
$$=re^{i\theta}.$$

$r=\sqrt{a^2+b^2}$ (modulus)

$\theta\in[0,2\pi)$ (argument)

Inner Products

Definition - A complex inner product on a vector space V is a mapping $\langle\cdot,\cdot\rangle: V\times V\to\mathbb{C}$ satisfying

1. $c\langle\vec{u},\vec{v}\rangle=\langle c\vec{u},\vec{v}\rangle=\langle\vec{u},\bar{c}\vec{v}\rangle$, $c\in\mathbb{C}$, $\vec{u},\vec{v}\in V$.
2. $\langle\vec{u}+\vec{v},\vec{w}\rangle=\langle\vec{u},\vec{w}\rangle+\langle\vec{v},\vec{w}\rangle$, $\vec{u},\vec{v},\vec{w}\in V$
3. $\langle\vec{u},\vec{v}\rangle=\langle\vec{v},\vec{u}\rangle$, $\vec{u},\vec{v}\in V$
4. $\langle\vec{u},\vec{u}\rangle=\|\vec{u}\|^2\in\mathbb{R}$, $\langle\vec{u},\vec{u}\rangle\geq 0$ with equality if and only if $\vec{u}=0$.

Example:

1. Let $\vec{u},\vec{v}\in\mathbb{C}^n$. The dot product defined by
$$\vec{u}\cdot\vec{v}=\langle\vec{u},\vec{v}\rangle=\bar{u}_1v_1+\bar{u}_2v_2+\dots+\bar{u}_nv_n$$
is an inner product.

2. Let $f,g\in C([-\pi,\pi];\mathbb{R})$
$$\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)g(x)\,dx$$
is a real valued inner product.

Definition - The angle between $\vec{u}, \vec{v} \in V$ is defined by

$$\theta = \arccos\left(\frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

and we say $\vec{u}, \vec{v}$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$.

Definition - A set of vectors $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is orthogonal if $\langle \vec{u}_i, \vec{u}_j \rangle = 0$ when $i \neq j$ and each $\vec{u}_i$ is nonzero. The set is orthonormal if, in addition to being orthogonal, $\|\vec{u}_i\| = 1$.

Example:

If $\{\vec{v}_1, \ldots, \vec{v}_n\} = \beta$ is an orthonormal basis then in the β basis

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \ldots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$$

coordinates

proof:

$$\vec{u} = a_1 \vec{v}_1 + \ldots + a_n \vec{v}_n$$

$$\Rightarrow \langle \vec{u}, \vec{v}_i \rangle = \langle a_1 \vec{v}_1 + \ldots + a_n \vec{v}_n, \vec{v}_i \rangle$$

$$= a_1 \langle \vec{v}_1, \vec{v}_i \rangle + \ldots + a_n \langle \vec{v}_n, \vec{v}_i \rangle$$

$$= a_i \langle \vec{v}_i, \vec{v}_i \rangle$$

$$= a_i.$$

Example:

Let $S^\perp = \{\vec{v} \in \mathbb{C}^n : \langle \vec{u}, \vec{v} \rangle = 0 \text{ for all } \vec{u} \in S\}$, which is called the orthogonal complement, is a subspace.

proof

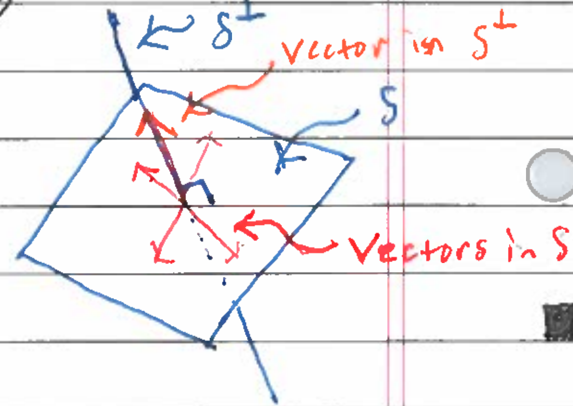
1. Since $\langle 0, \vec{u} \rangle = 0$ for all $\vec{u} \in S$ it follows that $0 \in S^\perp$
2. Let $\vec{v}, \vec{w} \in S^\perp$ and $\vec{u} \in S$. Therefore,

$$\langle \vec{v} + \vec{w}, \vec{u} \rangle = \langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle = 0$$

3. Let $\lambda \in \mathbb{C}$, $\vec{v} \in S^{\perp}$ and $\vec{u} \in S$. Therefore,

$$\langle \lambda \vec{v}, \vec{u} \rangle = \lambda \langle \vec{v}, \vec{u} \rangle = 0$$

and thus $\lambda \vec{v} \in S^{\perp}$.

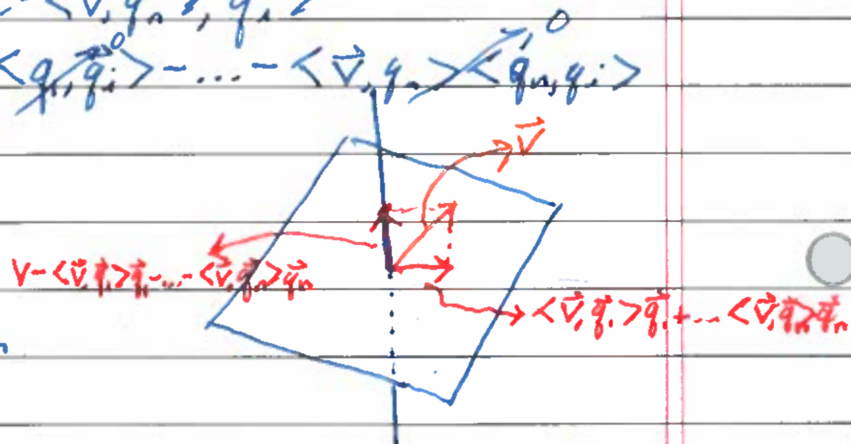


By items 1-3, $S^{\perp}$ is a subspace.

Corollary - Suppose $S = \{\vec{q}_1, \dots, \vec{q}_n\}$ is an orthonormal set, and $\vec{v} \in V$ the vector $\vec{r} = \vec{v} - \langle \vec{v}, \vec{q}_1 \rangle \vec{q}_1 - \dots \langle \vec{v}, \vec{q}_n \rangle \in S^{\perp}$.

proof:

$$\begin{aligned} \langle \vec{r}, \vec{q}_i \rangle &= \langle \vec{v} - \langle \vec{v}, \vec{q}_1 \rangle q_1 - \dots - \langle \vec{v}, \vec{q}_n \rangle, q_i \rangle \\ &= \langle \vec{v}, \vec{q}_i \rangle - \langle \vec{v}, \vec{q}_i \rangle \cancelto{0}{\langle q_1, \vec{q}_i \rangle} - \dots - \cancelto{0}{\langle \vec{v}, q_n \rangle \langle q_n, q_i \rangle} \\ &= \langle \vec{v}, \vec{q}_i \rangle - \langle \vec{v}, \vec{q}_i \rangle \\ &= 0 \end{aligned}$$



Gram-Schmidt Orthogonalization

$\beta = span\{\vec{v}_1, \dots, \vec{v}_n\}$

Find an orthonormal set $\{\vec{u}_1, \dots, \vec{u}_n\}$ such that $span\{\vec{u}_1, \dots, \vec{u}_n\} = span\{\vec{v}_1, \dots, \vec{v}_n\}$.

1. $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

2. $\vec{w}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle$ → Orthogonal compliment

   $\vec{u}_2 = \dfrac{\vec{w}_2}{\|\vec{w}_2\|}$ → normalize.

3. $\vec{w}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle - \langle \vec{v}_2, \vec{u}_1 \rangle$

   $\vec{u}_3 = \dfrac{\vec{w}_3}{\|\vec{w}_3\|}$.

   ⋮

Keep going until you are done.

Example:

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Find orthonormal vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ such that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$w_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot 3 \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ -3/4 \\ 1/4 \end{bmatrix}$$

$$\Rightarrow \|w_2\| = (1/16 + 1/16 + 9/16 + 1/16)^{1/2} = (12/16)^{1/2} = (3/4)^{1/2} = \sqrt{3}/2$$

$$\Rightarrow \vec{u}_2 = \frac{2}{\sqrt{3}}\begin{bmatrix} 1/4 \\ 1/4 \\ -3/4 \\ 1/4 \end{bmatrix}$$

$$\vec{w}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2$$

$$\Rightarrow \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}.$$