MTH 352/652: Homework #11

Due Date: May 01, 2024

1 Problems for Everyone

1. Consider the following sequence of functions

$$\delta_n(x) = \begin{cases} n - n^2|x| & |x| < 1/n \\ 0 & \text{otherwise.} \end{cases}$$

   (a) On the same set of axes, sketch $\delta_1(x)$, $\delta_2(x)$, and $\delta_4(x)$.

   (b) Explain why

$$\lim_{n\to\infty} \delta_n(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}.$$

   (c) Prove that if $f \in C^2$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0).$$

   (d) Explain why

$$\lim_{n\to\infty} n^{-1/2}\delta_n(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}.$$

   and prove that if $f \in C^2$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} n^{-1/2}\delta_n(x) f(x) dx = 0.$$

2. Evaluate the following integrals

   (a) $\displaystyle\int_{-\pi}^{\pi} \delta(x)\cos(x) dx.$

   (b) $\displaystyle\int_{1}^{2} \delta(x)(x-2) dx.$

   (c) $\displaystyle\int_{0}^{1} \delta\left(x - \frac{1}{3}\right) x^2 dx.$

   (d) $\displaystyle\int_{-1}^{1} \frac{\delta(x+2)}{1+x^2} dx.$

3. pg. 227, #6.1.4(a,c,d), #6.1.5(a,c).

4. pg. 240, #6.2.1.

Homework #11

#1

Consider the following sequence of functions

$$\delta_n(x) = \begin{cases} n - n^2|x|, & |x| < 1/n \\ 0, & \text{o.w} \end{cases}$$

(a) On the same set of axes $\delta_1(x)$, $\delta_2(x)$, and $\delta_4(x)$

(b) Explain why

$$\lim_{n\to\infty} \delta_n(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ \infty \text{ if } x = 0 \end{cases}$$

(c) Prove that if $f \in C^2$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx = f(0).$$

(d) Explain why

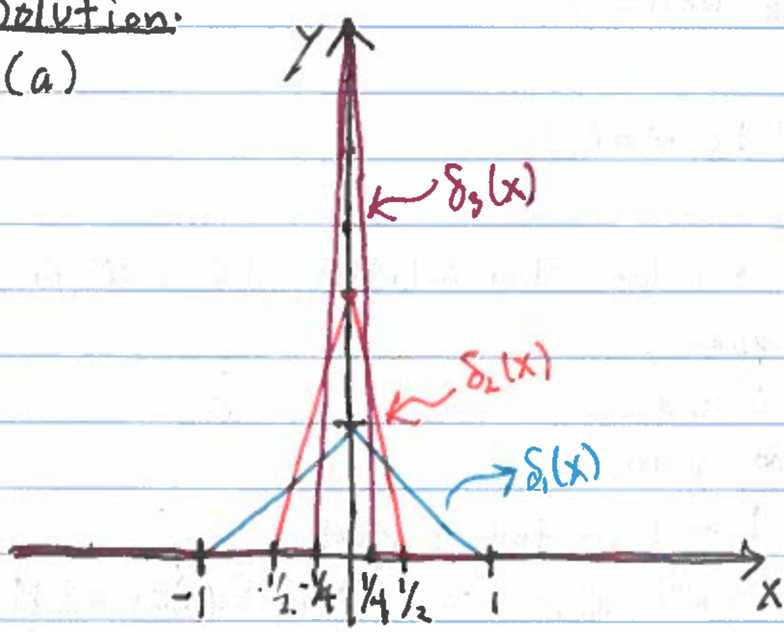
$$\lim_{n\to\infty} n^{-1/2} \delta_n(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ \infty \text{ if } x = 0 \end{cases}$$

and prove that if $f \in C^2$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} n^{-1/2} \delta_n(x) f(x)\,dx = 0.$$

Solution:

(a)



(b) If $x \neq 0$ and $n > 1/|x|$ it follows that $\delta_n(x) = 0$. If $x = 0$ it follows that $\delta_n(x) = n$. Consequently,

$$\lim_{n\to\infty} \delta_n(x) = \begin{cases} 0, & x \neq 0. \\ \infty, & x = 0 \end{cases}$$

(c) Computing we have that

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx - f(0) \right| &= \left| \int_{-1/n}^{1/n} f(x)\delta_n(x)\,dx - f(0) \right| \\
&= \left| \int_{-1/n}^{1/n} f(x)\delta_n(x)\,dx - \int_{-1/n}^{1/n} f(0)\delta_n(x)\,dx \right| \\
&= \left| \int_{-1/n}^{1/n} (f(x) - f(0))\delta_n(x)\,dx \right| \\
&\leq \int_{-1/n}^{1/n} |f(x) - f(0)|\,\delta_n(x)\,dx \\
&\leq \int_{-1/n}^{1/n} M \cdot |x|\,\delta_n(x)\,dx,
\end{aligned}$$

where $M = \max_{-1 \leq x \leq 1} \{f(x)\}$. Therefore,

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx - f(0) \right| &\leq 2\int_{0}^{1/n} Mx \cdot (n - n^2 x)\,dx \\
&= 2M \cdot \left(\frac{n}{2n^2} - \frac{n^2}{3n^3} \right) \\
&= \frac{M}{3n}.
\end{aligned}$$

Consequently, by the squeeze theorem

$$\lim_{n\to\infty} \left| \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx - f(0) \right| = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx = f(0).$$

(d) If $x \neq 0$ and $n > 1/|x|$ it follows that $\delta_n(x) = 0$. If $x = 0$ it follows that $\delta_n(x) = n$. Consequently,

$$\lim_{n\to\infty} n^{-1/2}\delta_n(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0. \end{cases}$$

Now, letting $M = \max_{-1 \leq x \leq 1} |f(x)|$ it follows that

$$\left| \int_{-\infty}^{\infty} n^{-1/2} f(x)\delta_n(x)\,dx \right| \leq \int_{-1/n}^{1/n} n^{-1/2} |f(x)|\,\delta_n(x)\,dx \leq 2\int_{0}^{1/n} M n^{-1/2} \delta_n(x)\,dx = \frac{2M}{n^{1/2}}.$$

Therefore, by the squeeze theorem

$$\lim_{n\to\infty} \left| \int_{-\infty}^{\infty} n^{-1/2} f(x)\,\delta_n(x)\,dx \right| = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_{-\infty}^{\infty} n^{-1/2} f(x)\,\delta_n(x)\,dx = 0$$

■

#2

Evaluate the following integrals

(a) $\int_{-\pi}^{\pi} \delta(x)\cos(x)\,dx$

(b) $\int_{1}^{2} \delta(x)(x-2)\,dx$

(c) $\int_{0}^{1} \delta(x-1/3)\,x^2\,dx$

(d) $\int_{-1}^{1} \delta(x+2)/1+x^2\,dx$

Solution:

(a) $\int_{-\pi}^{\pi} \delta(x)\cos(x)\,dx = \cos(0) = 1.$

(b) $\int_{1}^{2} \delta(x)(x-2)\,dx = 0$, since $0 \notin [1,2]$.

(c) $\int_{0}^{1} \delta(x-1/3)\,x^2\,dx = 1/9.$

(d) $\int_{-1}^{1} \delta(x+2)/1+x^2\,dx = 0$, since $-2 \notin [-1,1]$.

■

pg. 227, #6.1.4 (a,c,d)

Find and sketch a graph of the derivative of the following functions

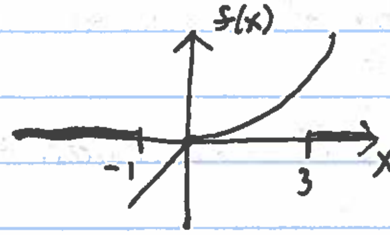
(a) $f(x) = \begin{cases} x^2, & 0<x<3 \\ x, & -1<x<0 \\ 0, & \text{o.w.} \end{cases}$

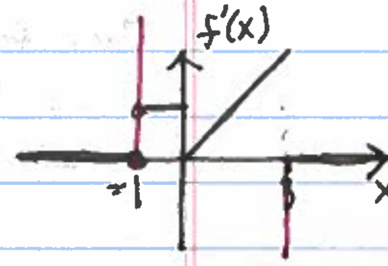
(c) $h(x) = \begin{cases} \sin(\pi x), & x \geq 1 \\ 1-x^2, & -1 \leq x \leq 1 \\ e^x, & x<-1 \end{cases}$

(d) $k(x) = \begin{cases} \sin(x), & x<-\pi \\ x^2-\pi^2, & -\pi<x<0 \\ e^{-x}, & x>0 \end{cases}$

Solution:

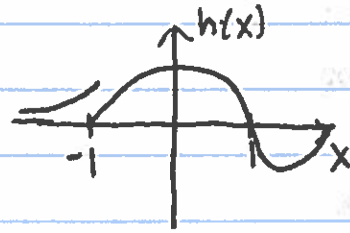
(a) $f(x)=\begin{cases} x^2, & 0<x<3 \\ x, & -1<x<0 \\ 0, & \text{o.w} \end{cases}$

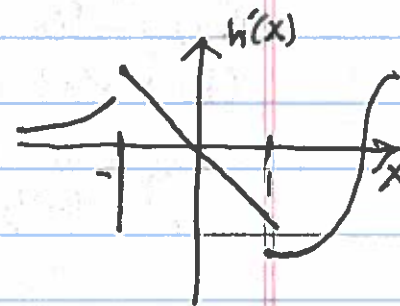




$$\Rightarrow f'(x)=-9\,\delta(x-3)-\delta(x+1)+\begin{cases} 2x, & 0<x<3 \\ 1, & -1<x<0 \\ 0, & \text{o.w} \end{cases}$$

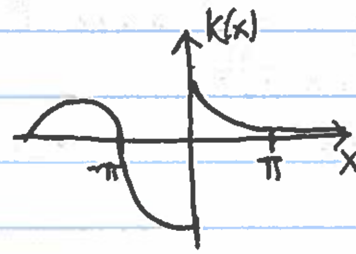
(c) $h(x)=\begin{cases} \sin(\pi x), & x>1 \\ 1-x^2, & -1<x<1 \\ e^x, & x<-1 \end{cases}$

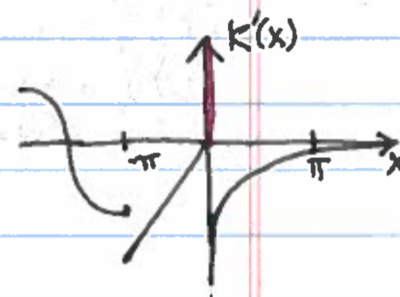




$$\Rightarrow f'(x)=-e^{-1}\,\delta(x+1)+\begin{cases} \pi\cos(\pi x), & x>1 \\ -2x, & -1<x<1 \\ e^x, & x<-1 \end{cases}$$

(d) $k(x)=\begin{cases} \sin(x), & x<-\pi \\ x^2-\pi^2, & -\pi<x<0 \\ e^{-x}, & x>0 \end{cases}$





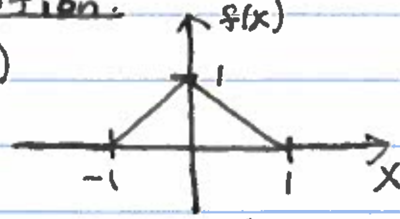
$$k'(x)=(1+\pi^2)\,\delta(x)+\begin{cases} \cos(x), & x<-\pi \\ 2x, & -\pi<x<0 \\ -e^{-x}, & x>0 \end{cases}$$

pg. 227, #6.1.5 (a,c)

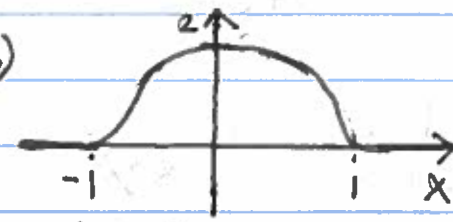
(a) $f(x)=\begin{cases} x+1, & -1<x<0 \\ 1-x, & 0<x<1 \\ 0, & \text{o.w.} \end{cases}$ (c) $s(x)=\begin{cases} 1+\cos(\pi x), & -1<x<1 \\ 0, & \text{o.w.} \end{cases}$

Solution:

(a)



(b)



$\Rightarrow f'(x)=\begin{cases} 1, & -1<x<0 \\ -1, & 0<x<1 \\ & \text{o.w.} \end{cases}$ $\Rightarrow f'(x)=\begin{cases} -\pi\sin(\pi x), & -1<x<1 \\ 0, & \text{o.w} \end{cases}$

$\Rightarrow f''(x)=-2\delta(x)+\delta(x-1)+\delta(x+1) \Rightarrow f''(x)=\begin{cases} -\pi^2\cos(\pi x), & -1<x<1 \\ 0, & \text{o.w.} \end{cases}$

■

pg. 241, #6.2.1

Let $c>0$. Find the Green's function for the boundary value problem $-cu''=f(x)$, $u(0)=0$, $u'(1)=0$. Write down the general form of the solution and verify the integral formula by differentiation.

Solution:

Setting $-cG''=\delta(x-y)$, we have that

$$G(x,y)=\begin{cases} Ax+B, & x<y \\ Cx+D, & x>y \end{cases}$$

Boundary conditions imply that

$G(0,y)=0 \Rightarrow B=0$

$G'(1,y)=0 \Rightarrow C=0.$

Therefore,

$$G(x,y) = \begin{cases} Ax, & x<y \\ D, & x>y \end{cases}$$

Continuity implies

$$Ay = D$$

Finally, the jump condition implies

$$\lim_{x\to y^+} G'(x,y) - \lim_{x\to y^-} G'(x,y) = \frac{-1}{c}$$

$$\Rightarrow \quad 0 - A = -1/c$$

$$\Rightarrow \quad A = 1/c$$

Therefore,

$$G(x,y) = \begin{cases} \frac{1}{c}x, & x<y \\ \frac{1}{c}y, & x>y \end{cases}$$

Consequently,

$$u(x) = \int_0^x \frac{1}{c} y f(y)\,dy + \int_x^1 \frac{1}{c} x f(y)\,dy$$

Therefore,

$$u'(x) = \cancel{\frac{1}{c} x f(x)} + \int_x^1 \frac{1}{c} f(y)\,dy - \cancel{\frac{1}{c} x f(x)}$$

$$u''(x) = -\frac{1}{c} f(x)$$

∎