MTH 352/652: Homework #1

Due Date: January 26, 2024

1 Problems for Everyone

1. Sign up for Piazza. I will check the roster for your name.
2. Determine which of the following operators are linear. If an operator is linear, prove it. If an operator is nonlinear find a counterexample, i.e., two functions u, v for which $L[u+v] \neq L[u] + L[v]$ or a function u for which $L[cu] \neq cL[u]$.
   (a) $L[u] = u_x + xu_t$
   (b) $L[u] = u_x + uu_t$
   (c) $L[u] = u_x + u_t^2$
   (d) $L[u] = u_x + u_t + 1$
   (e) $L[u] = \sqrt{1+x^2}\cos(t)u_x + u_{txt} - \arctan(x/y)u$
3. For each of the following equations, state the order and whether it is nonlinear, linear homogenous, or linear inhomogeneous. There is no need prove anything for this problem.
   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $u_x(1+u_x^2)^{-1/2} + u_t(1+u_t^2)^{-1/2} = 0$
   (f) $u_x + e^t u_t = 0$
   (g) $u_t + u_{xxxx} + \sqrt{1+u} = 0$
4. Verify that $u(x,y) = f(x)g(y)$ is a solution to the PDE $uu_{xy} = u_x u_y$ for all pairs of differentiable functions f and g of one variable.
5. Show that the following functions solve the PDE $u_{xx} + u_{yy} = 0$:
   (a) $u(x,y) = e^x \cos(y)$
   (b) $u(x,y) = 1 + x^2 - y^2$
   (c) $u(x,y) = x^3 - 3xy^2$
   (d) $u(x,y) = \ln(x^2 + y^2)$
   (e) $u(x,y) = \arctan(y/x)$
   (f) $u(x,y) = \dfrac{x}{x^2+y^2}$
   (g) $u(x,y) = \sin(nx)(e^{ny} - e^{-ny})$, $n \in \mathbb{R}$

6. Solve the PDE $3u_t + u_{xt} = 0$. **Hint:** Let $v = u_t$.

7. Find the general solution to the PDE $u_{xy} = 0$ in terms of two arbitrary functions.

8. Find a function $u(t, x)$ that satisfies the PDE

$$u_{xx} = 0$$

on the domain $0 < x < 1$, $t > 0$ subject to the boundary conditions $u(t, 0) = t^2$ and $u(t, 1) = 1$.

9. Show that the nonlinear equation $u_t = u_x^2 + u_{xx}$ can be reduced to the linear equation $w_t = w_{xx}$ by changing the variable to $w = e^u$.

10. Solve the following initial value problems and graph the solutions at times $t = 1, 2$ and 3.

(a) $u_t - 3u_x = 0$ with $u(0, x) = e^{-x^2}$

(b) $2u_t + 3u_x = 0$ with $u(0, x) = \sin(x)$

(c) $u_t + 2u_x = 0$ with $u(-1, x) = x/(1 + x^2)$

(d) $u_t + u_x + u = 0$ with $u(0, x) = \arctan(x)$

Homework #1

#4

Verify that $u(x,y) = f(x)g(y)$ is a solution to the PDE

$$u u_{xy} = u_x u_y$$

for all pairs of differentiable functions f and g of one variable.

Solution:

Substituting, we have that

$$u_x = f'(x)\,g(y),\quad u_y = f(x)g'(y),\quad u_{xy} = f'(x)g'(y)$$

Therefore,

$$u u_{xy} = f(x)g(y)f'(x)g'(x)$$

$$u_x u_y = f'(x)g(y)f(x)g'(y)$$

$$\Rightarrow u u_{xy} = u_x u_y$$

$\blacksquare$

#6.

Solve the PDE $3u_t + u_{xt} = 0$.

Solution:

Let $v = u_t$. Therefore,

$$3v + v_x = 0$$

$$\Rightarrow v_x = -3v$$

$$\Rightarrow v = f(t)e^{-3x}$$

$$\Rightarrow u_t = f(t)\,e^{-3x}$$

$$\Rightarrow u = e^{-3x}\int_0^t f(s)ds + g(x),$$

where f, g are arbitrary

$\blacksquare$

#8.

Find a function $u(t,x)$ that satisfies

$$u_{xx}=0$$

on the domain $0<x<1$, $t>0$ subject to the boundary conditions

$$u(t,0)=t^2 \text{ and } u(t,1)=1.$$

Solution:

Solving, we have that

$$u_x = f(t)$$

$$\Rightarrow u = f(t)x + g(t),$$

where f and g are arbitrary. Consequently,

$$u(t,0)=t^2=g(t)$$

$$\Rightarrow u(t,1)=1=f(t)+t^2$$

$$\Rightarrow f(t)=1-t^2.$$

Therefore,

$$u(t,x)=(1-t^2)x+t^2.$$

#9

Show that the nonlinear equation $u_t=u_x^2+u_{xx}$ can be reduced to the linear equation $w_t=w_{xx}$ by changing the variable to $w=e^u$.

Solution:

Differentiating, we have that

$$w_t = u_t e^u = (u_x^2+u_{xx})e^u$$

$$w_x = u_x e^u$$

$$w_{xx} = u_{xx}e^u + u_x^2 e^u = (u_x^2+u_{xx})e^u.$$

Therefore,

$$w_t = w_{xx}.$$

#10

Solve the following initial value problems and graph the Solutions at times $t=1, 2$, and 3.

(a) $u_t - 3u_x = 0$, $u(0,x) = e^{-x^2}$

(b) $2u_t + 3u_x = 0$, $u(0,x) = \sin(x)$

(c) $u_t + 2u_x = 0$, $u(-1,x) = x/1+x^2$

(d) $u_t + u_x + u = 0$, $u(0,x) = \tan^{-1}(x)$

Solution:

(a) Letting $u(t,x) = f(x+3t)$ we have that

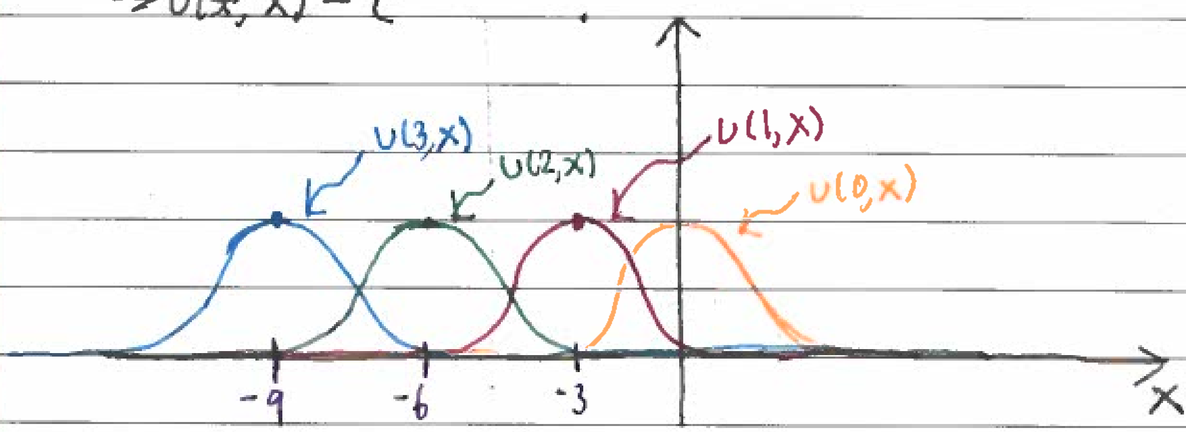
$$u_t = 3f'(x+3t)$$
$$u_x = f'(x+3t)$$

Consequently,

$$u(t,x) = f(x+3t)$$

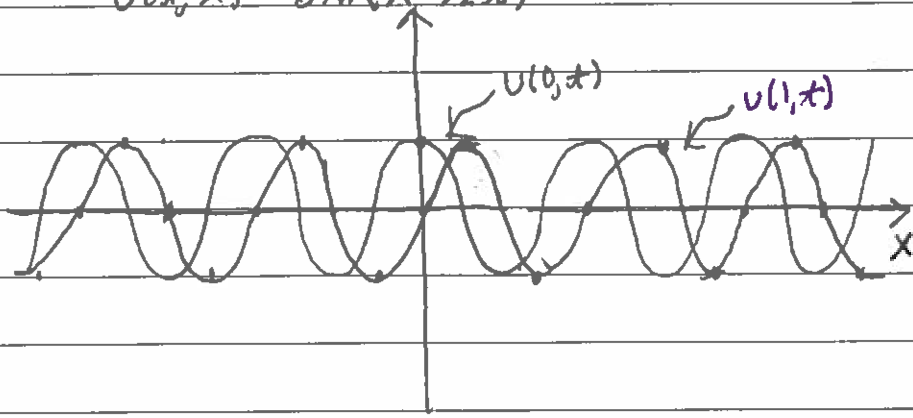
is a solution. Applying initial conditions we have that

$$u(0,x) = f(x) = e^{-x^2}$$
$$\Rightarrow u(t,x) = e^{-(x+3t)^2}.$$



(b) The solution is given by

$$u(t,x) = \sin(x - \tfrac{3}{2}t)$$



(c) Let $\tau = 1 + t$. Therefore, we have that $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t}$ and thus

$$u_\tau + 2u_x = 0$$
$$\Rightarrow u(\tau, x) = f(x - 2\tau)$$
$$\Rightarrow u(0,x) = f(x) = x/1+x^2$$
$$\Rightarrow u(\tau, x) = \frac{x - 2\tau}{1 + (x - 2\tau)^2}$$
$$\Rightarrow u(t,x) = \frac{x - 2(1+t)}{1 + (x - 2(1+t))^2}$$

(d) Let $z = x - t$ and $\tau = t$. Therefore,

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial z}$$
$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -\frac{\partial}{\partial z} + \frac{\partial}{\partial \tau}.$$

Therefore

$$u_t + u_x + u = u_\tau + u = 0$$
$$\Rightarrow u(\tau, z) = f(z)e^{-\tau}$$
$$\Rightarrow u(t, x) = f(x - t)e^{-t}$$

Applying initial conditions we have that

$$u(t,x) = \tan^{-1}(x - t)e^{-t}$$