MTH 352/652: Homework #2

Due Date: February 02, 2024

1 Problems for Everyone

1. Let $c \neq 0$ and suppose $u(t,x)$ solves the following initial value problem

$$u_t + cu_x = 0 \text{ and } u(0,x) = f(x).$$

Suppose f is continuous and satisfies $\lim_{|x|\to\infty} f(x) = 0$. Prove that $\lim_{t\to\infty} u(t,x) = 0$.

2. Solve the following initial value problems in the region $x \in \mathbb{R}$, $t > 0$

   (a) $u_t + xtu_x = 0$ and $u(0,x) = f(x)$

   (b) $u_t + xu_x = e^t$ and $u(0,x) = f(x)$

3. Solve the following initial value problems in the region $x \in \mathbb{R}$, $t > 0$

   (a) $u_t + xu_x = -tu$ and $u(0,x) = f(x)$

   (b) $tu_t + xu_x = -2u$ and $u(0,x) = f(x)$

   (c) $u_t + u_x = -tu$ and $u(0,x) = f(x)$

4. Consider the following initial value problems in the region $x \in \mathbb{R}$, $t > 0$:

$$u_t + u_x + u^2 = 0 \text{ and } u(0,x) = f(x).$$

   (a) Find the general solution to this initial value problem.

   (b) Show that if $f(x)$ is bounded and positive, i.e., $0 \leq f(x) \leq M$, then the solution exists for all $t > 0$ and

$$\lim_{t\to\infty} u(t,x) = 0.$$

   (c) Show that if $f(x)$ is negative, so $f(x) < 0$ at some $x \in \mathbb{R}$, then the solution blows up in finite time:

$$\lim_{t\to\tau^-} u(t,y) = -\infty$$

   for some $\tau > 0$ and some $y \in \mathbb{R}$.

5. Consider the equation

$$u_t + xu_x = 0$$

with the boundary condition $u(t,0) = \phi(t)$.

   (a) For $\phi(t) = t$, show that no solution exists.

   (b) For $\phi(t) = 1$, show that there are infinitely many solutions.

Homework #2

#1

Let $c \neq 0$ and suppose $u(t,x)$ solves the following initial value problem

$$u_t + cu_x = 0, \quad u(0,x) = f(x).$$

Suppose f is continuous and satisfies $\lim_{|x|\to\infty} f(x) = 0$. Prove that $\lim_{t\to\infty} u(t,x) = 0$.

Solution:

The solution to the initial value problem is $u(t,x) = f(x-ct)$.

Therefore,

$$\lim_{t\to\infty} u(t,x) = \lim_{t\to\infty} f(x-ct) = f\left(\lim_{t\to\infty}(x-ct)\right) = f(\pm\infty) = 0.$$

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#2

Solve the following initial value problems in the region $x \in \mathbb{R}$, $t > 0$.

(a) $u_t + xtu_x = 0$, $u(0,x) = f(x)$

(b) $u_t + xu_x = e^t$, $u(0,x) = f(x)$.

Solution:

(a) The characterestic curves satisfy

$$\frac{dx}{dt} = xt$$

$$\Rightarrow \ln(x) = \frac{t^2}{2} + c$$

$$\Rightarrow x = ce^{t^2/2}$$

Let $z = \ln(|x|) - t^2/2$ and $\tau = t$. Therefore,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = -t\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \tau}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial u}{\partial z}$$

Substituting in we obtain

$$u_t + xtu_x = -t u_z + u_\tau + xt \cdot \frac{1}{x} u_z = 0$$

$$\Rightarrow u_\tau = 0$$

$$\Rightarrow u(\tau, z) = g(z)$$

$$\Rightarrow u(t, x) = g(\ln|x| - t^2/2)$$

Therefore,

$$u(0, x) = g(\ln(|x|) = f(x)$$

$$\Rightarrow g(x) = f(e^x).$$

$$\Rightarrow u(t, x) = f(e^{\ln|x| - t^2/2}),$$

The solution is therefore

$$u(t, x) = f(xe^{-t^2/2}).$$

(b). The characteristic curves satisfy

$$\frac{dx}{dt} = x$$

$$\Rightarrow \ln(|x|) - t = C$$

Let $z = \ln(|x|) - t$ and $\tau = t$. Consequently,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \tau}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial u}{\partial z}.$$

Therefore,

$$u_t + xu_x = e^t \Rightarrow u_\tau = e^\tau.$$

Consequently,

$$u(\tau, z) = e^\tau + g(z)$$

$$\Rightarrow u(t, x) = e^t + g(\ln(|x| - t)$$

$$\Rightarrow u(0, x) = 1 + g(\ln(|x|) = f(x)$$

$$\Rightarrow g(\ln(|x|) = f(x) - 1.$$

$$\Rightarrow g(x) = f(e^x) - 1.$$

The solution is therefore,

$$U(t,x) = e^{t} + f(e^{\ln(|x|)-t}) - 1$$

$$\Rightarrow U(t,x) = e^{t} - 1 + f(xe^{-t}).$$

$\blacksquare$

#3

Solve the following initial value problems in the region $x \in \mathbb{R}, t > 0$.

(a) $U_t + xU_x = -tU,\ U(0,x) = f(x)$

(b) $tU_t + xU_x = -2U,\ U(0,x) = f(x)$

(c) $U_t + U_x = -tU,\ U(0,x) = f(x).$

Solution:

(a) Let $z = \ln(|x|) - t,\ \tau = t$. Therefore, we have that

$$U_\tau = -\tau U$$

$$\Rightarrow U = g(z)e^{-\tau^2/2}$$

Consequently,

$$U(t,x) = g(\ln(|x|) - t)e^{-t^2/2}$$

$$\Rightarrow U(0,x) = g(\ln(|x|)) = f(x)$$

$$\Rightarrow g(x) = f(e^{x}).$$

Therefore,

$$U(t,x) = f(xe^{-t})e^{-t^2/2}.$$

(b) The characteristic curves satisfy

$$\frac{dx}{dt} = \frac{x}{t}$$

$$\Rightarrow \ln(|x|) - \ln(|t|) = c$$

Let $z = \ln(|x|) - \ln(|t|),\ \tau = t$. Therefore,

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{1}{t}\frac{\partial U}{\partial z} + \frac{\partial U}{\partial \tau}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial v}{\partial z}$$

Therefore,

$$t v_t + x v_x = -2v \Rightarrow \tau v_\tau = -2v$$

and thus

$$\frac{1}{v} v_\tau = \frac{-2}{\tau}$$

$$\Rightarrow \ln(|v|) = -2\ln(\tau) + g(z)$$

$$\Rightarrow v = \frac{g(z)}{\tau^2}.$$

$$\Rightarrow v(t,x) = \frac{g(\ln(|x|-\ln(|t|))}{t^2}$$

*This will not work out as written.

(c) Letting $z = x - t$, $\tau = t$ we have that

$$v_\tau = -\tau v$$

$$\Rightarrow v(\tau, z) = g(z) e^{-\tau^2/2}$$

$$\Rightarrow v(t,x) = f(t-x) e^{-t^2/2}$$

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#4

Consider the following initial value problem in the region $x \in \mathbb{R}$, $t > 0$

$$v_t + v_x + v^2 = 0, \quad v(0,x) = f(x).$$

(a) Letting $z = x - t$, $\tau = t$ we have that

$$v_\tau = -v^2$$

$$\Rightarrow -\frac{1}{v^2} v_\tau = 1$$

$$\Rightarrow \frac{1}{v} = \tau + g(z)$$

$$\Rightarrow v = \frac{1}{\tau + g(z)}$$

$$\Rightarrow u(t,x) = \frac{1}{t + g(x-t)}$$

$$\Rightarrow u(0,x) = \frac{1}{g(x)} = f(x).$$

Therefore,

$$u(t,x) = \frac{1}{t + 1/f(x-t)} = \frac{f(x-t)}{1 + t f(x-t)}.$$

(b) If f is bounded and positive then for $t > 0$,

$$u(t,x) = \frac{f(x-t)}{1 + t f(x-t)} \leq \frac{f(x-t)}{t f(x-t)} = \frac{1}{t}.$$

Therefore, by the squeeze theorem

$$\lim_{t \to \infty} u(t,x) = 0$$

(c) Solving when the denominator is 0, we have

$$1 + t^* f(x - t^*) = 0$$

$$\Rightarrow t^* f(x - t^*) = -1$$

Letting $z^* = x - t^*$ we have that

$$(x - z^*) f(z^*) = -1$$

$$\Rightarrow f(z^*) = -\frac{1}{x - z^*}$$

Since $\lim_{z^* \to x} -\frac{1}{x - z^*} = -\infty$ we can find z^* so that this is an equality.

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#5

Consider the equation

$$U_t + xU_x = 0$$

with the boundary condition $u(t,0) = \phi(t)$.

(a) For $\phi(t) = t$, show that no solution exists.

(b) For $\phi(t) = 1$, show that there are infinitely many solutions.

Solution:

(a) From problem #2(a) we know that the generic solution is constant along the characteristic curves

$$x(t) = ce^{t^2/2}$$

$$\Rightarrow u(t,x) = f(xe^{-t^2/2})$$

where f is any arbitrary function. Applying boundary conditions we have:

$$f(0) = t$$

which is not possible.

(b) Applying boundary conditions, we now have

$$f(0) = 1$$

which has infinite number of solutions.