MTH 352/652: Homework #3

Due Date: February 09, 2024

1 Problems for Everyone

2 1. By changing variables to $\xi = x - ct$ and $\eta = x + ct$, show that the wave equation $u_{tt} = c^2 u_{xx}$ is equivalent to the following PDE:

$$u_{\xi\eta} = 0.$$

1 2. Consider the following initial value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \ x \in \mathbb{R}, \ t > 0,$$
$$u(0,x) = \frac{4}{4+x^2} \text{ and } u_t(0,x) = 0,$$

where $c > 0$ is a constant.

(a) What do the initial conditions model in this situation?
(b) Calculate the exact solution $u(t,x)$ to this problem.
(c) Assuming $c = 1$, plot $u(t,x)$ at $t = 0$, $t = 1$, and $t = 2$.
(d) Calculate $\lim_{t\to\infty} u(t,x)$.
(e) Discuss the qualitative behavior of the solution.

1 3. Consider the following initial value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \ x \in \mathbb{R}, \ t > 0,$$
$$u(0,x) = 0 \text{ and } u_t(0,x) = \frac{4}{4+x^2},$$

where $c > 0$ is a constant.

(a) What do the initial conditions model in this situation?
(b) Calculate the exact solution $u(t,x)$ to this problem.
(c) Assuming $c = 1$, plot $u(t,x)$ at $t = 0$, $t = 1$, and $t = 2$.
(d) Calculate $\lim_{t\to\infty} u(t,x)$.
(e) Contrast how the solution to this initial value problem is different from the solution to problem #1.

4. pg. 60-62, #2.4.1-#2.4.3

2 5. The *Poisson-Darboux* equation is

$$u_{tt} - u_{xx} - \frac{2}{x}u_x = 0.$$

By making the substitution, $w = xu$, solve the Poisson-Darboux equation with the initial data $u(0,x) = 0$, $u_t(0,x) = g(x)$ where $g(x) = g(-x)$, i.e., g is an even function.

2 6. For a classical solution $u(t,x)$ to of the unit speed wave equation $u_{tt} = u_{xx}$ the energy and momentum density are defined by

$$E(t,x) = \frac{1}{2}\left(u_t^2 + u_x^2\right) \text{ and } P(t,x) = u_t u_x,$$

respectively.

(a) Show that $E_t = P_x$ and $E_x = P_t$.

(b) Show that both $E(t,x)$ and $P(t,x)$ satisfy the wave equation.

~~(c)~~ Show that

$$\int_{-\infty}^{\infty} u(t,x)dx$$

is constant in time.

2 7. A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where $r = \sqrt{x^2+y^2+z^2}$ is the distance from the origin. The wave equation in this case takes the form:

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$$
$$u(0,r) = \phi(r),$$
$$u_t(0,r) = \psi(r)$$

where $\phi(r), \psi(r)$ are both even functions of r. By making the change of variables $v = ru$, solve the spherical wave equation. What does your solution tell you about the propagation of spherical waves in three dimensions?

8. Solve the equation $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ with the initial conditions $u(0,x) = e^{-x^2}$ and $u_t(0,x) = \frac{1}{1+x^2}$. **Hint:** Factor the operator as we did for the wave equation.

2 Choose One of These Problems to Complete

1. In 1858 the first trans-Atlantic telegraph cable was laid down allowing direct communication between the United States and England. The first message sent by Queen Victoria to President James Buchanan contained 98 words and took 16 hours to communicate. This was an incredible improvement over the ten days it took to communicate by ships! However, within a month of its first operation Dr. Wildman Whitehouse (a medical doctor with no real training in electrical engineering) essentially melted the cable by applying too high of a voltage. The reaction from the media to this announcement was volatile with many writers hinting that the line was a hoax. Nevertheless, engineers persisted and a working cable was installed in 1866 that could transmit a blazing eight words per minute!

   In this problem our goal is to mathematically understand how signals can be sent along wires. In particular, our goal is to tune specific electrical properties of the wire to arrive at the wave equation. As we saw in class, the wave equation is particularly nice for sending signals because the solutions retain the shape of the initial profile. That is, somebody receiving the signal would know precisely what the initial shape of the signal is.

   (a) We will model the wire as an infinitely long straight line of resistance $R \geq 0$, inductance $L \geq 0$, grounding resistance $G \geq 0$ and capacitance $C \geq 0$. Letting $I(t,x)$ and $V(t,x)$

denote the current and voltage across the wire at position x and time t, the partial differential equations satisfied by I and V are given by:

$$V_x = -RI - LI_t, \tag{1}$$
$$I_x = -GV - CV_t. \tag{2}$$

Show that I can be eliminated from these equations to yield the following equation:

$$LCV_{tt} + (LG + RC)V_t + RGV = V_{xx}. \tag{3}$$

(b) If we define $c = 1/\sqrt{LC}$, $a = c^2(LG + RC)$, and $b = c^2RG$ then the above equation becomes:

$$V_{tt} + aV_t + bV = c^2V_{xx}. \tag{4}$$

This equation is known as the **telegrapher's equation**. If $a = 0$ and $b = 0$ this equation becomes the wave equation. If $L > 0$ and $C > 0$ show that if $a = 0$ and $b = 0$ then both $R = 0$ and $G = 0$. Why is this an unrealistic case?

(c) By making a substitution of the form $V(t, x) = e^{-\lambda t}u(t, x)$ and by appropriately picking λ in terms of the constants a, b and c, show that the telegraphers equation can be reduced to the following partial differential equation:

$$u_{tt} + ku = c^2u_{xx}, \tag{5}$$

where k is a constant that depends on a, b and c.

(d) Show that $k = 0$ only when $RC = LG$.

(e) In the case when $RC = LG$ with the initial conditions $u(0, x) = f(x)$ and $u_t(0, t) = 0$, show that a solution to a solution to the telegrapher's equation is given by

$$V(t, x) = \frac{e^{-\lambda t}}{2}(f(x - ct) + f(x + ct)), \tag{6}$$

What does this solution mean in practical terms? In particular, why is it useful for sending signals?

Remark: This process of tuning the electrical parameters so the $RC = LG$ is called "Pupinizing" the cable after one of its discoverers Michael Pupin.

2. pg. 62, #2.4.15

Homework #3

#2

By changing variables to $\xi = x - ct$ and $\eta = x + ct$, show that the wave equation $u_{tt} = c^2 u_{xx}$ is equivalent to the following PDE:

$$u_{\xi\eta} = 0.$$

Solution:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\frac{\partial^2}{\partial \xi \partial \eta} + c^2\frac{\partial^2}{\partial \eta^2}$$

Therefore,

$$c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$\Rightarrow 4c^2 u_{\xi\eta} = 0$$

$$\Rightarrow u_{\xi\eta} = 0$$

$\blacksquare$

#2.

Consider the initial value problem for the wave equation:

$$u_{tt}=c^2u_{xx}\ ,\ x\in\mathbb{R},\ t>0$$
$$u(0,x)=\frac{4}{4+x^2}\ ,\ u_t(0,x)=0.$$

(b) Calculate the exact solution.

(d) Calculate $\lim_{t\to\infty} u(t,x)$.

Solution:

(b) $u(t,x)=\dfrac{2}{4+(x-ct)^2}+\dfrac{2}{4+(x+ct)^2}$.

(d) $\lim_{t\to\infty} u(t,x)=0$. ∎

#3.

Consider the following initial value problem for the wave equation

$$u_{tt}=c^2u_{xx},\ x\in\mathbb{R},\ t>0,$$
$$u(0,x)=0,\ u_t(0,x)=\frac{4}{4+x^2},$$

where $c>0$ is a constant.

(b) Calculate the exact solution $u(t,x)$ to this problem.

(d) Calculate $\lim_{t\to\infty} u(t,x)$.

Solution:

(b) $u(t,x)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct}\frac{4}{4+s^2}\,ds=\frac{1}{2c}\int_{x-ct}^{x+ct}\frac{1}{1+(s/2)^2}\,ds=\frac{1}{c}\int_{(x-ct)/2}^{(x+ct)/2}\frac{1}{1+u^2}\,du$

$\Rightarrow u(t,x)=\dfrac{1}{c}\tan^{-1}\left(\dfrac{x+ct}{2}\right)-\dfrac{1}{c}\tan^{-1}\left(\dfrac{x-ct}{2}\right)$

(c) $\lim u(t,x)=\pi/c$. ∎

#5.

The Poisson-Darboux equation is

$$u_{tt} - u_{xx} - \frac{2}{x} u_x = 0$$

By making the substitution, $w = xu$, solve the Poisson-Darboux equation with the initial data $u(0,x) = 0$, $u_t(x,0) = g(x)$ where $g(x)$ is an even function

Solution:

Let $w = xu$. Therefore,

$$w_{tt} = x u_{tt}$$

$$w_{xx} = 2u_x + x u_{xx}$$

Consequently,

$$w_{tt} - w_{xx} = x u_{tt} - 2u_x - x u_{xx} = x\left(u_{tt} - \frac{2}{x} u_x - u_{xx}\right) = 0$$

$$w(0,x) = 0$$

$$w_t(0,x) = x g(x).$$

Therefore,

$$w(t,x) = \frac{1}{2} \int_{x-t}^{x+t} s g(s)\, ds$$

$$\Rightarrow u(t,x) = \frac{1}{2x} \int_{x-t}^{x+t} s g(s)\, ds.$$

Note, by L'Hospitals rule we have that

$$u(t,0) = \lim_{x \to 0} \frac{1}{2x} \int_{x-t}^{x+t} s g(s)\, ds = \lim_{x \to 0} \frac{(x+t) g(x+t) - (x-t) g(x-t)}{2}$$

$$= \frac{t g(t) + t g(-t)}{2}$$

$$= g(t).$$

$\blacksquare$

#6.

For a classical solution $v(t,x)$ to the unit speed wave equation $v_{tt}=v_{xx}$ the energy and momentum density are defined by

$$E(t,x)=\frac{1}{2}(v_t^2+v_x^2) \quad \text{and} \quad P(t,x)=v_t v_x.$$

respectively.

(a) Show that $E_t=P_x$ and $E_x=P_t$

(b) Show that both $E(t,x)$ and $P(t,x)$ satisfy the wave equation.

Solution:

(a) $E_t=v_t v_{tt}+v_x v_{tx}$ and $P_x=v_{tx}v_x+v_t v_{xx}$

$\Rightarrow E_t=v_t(v_{xx})+v_x v_{tx}=P_x.$

$E_x=v_t v_{tx}+v_x v_{xx}$ and $P_t=v_{tt}v_x+v_t v_{tx}$

$\Rightarrow P_t=(v_{xx})v_x+v_t v_{tx}=E_x$

(b). Differentiating, we have that

$E_{tt}=P_{xt}=P_{tx}=E_{xx}$

$P_{tt}=E_{xt}=E_{tx}=P_{xx}.$

$\blacksquare$

#7.

The spherical wave equation takes the form

$$v_{tt}=c^2\left(v_{rr}+\frac{2}{r}v_r\right)$$

$$v(0,r)=\phi(r),\ v_t(0,r)=\psi(r).$$

where ϕ and ψ are even functions of r. By making the change of variables $V=rv$, solve the spherical wave equation. What does your solution tell you about the propagation of sperical waves in $\mathbb{R}^3$?

Solution:

Letting $v = ru$ we have

$$v_{tt} = r u_{tt}$$

$$v_{rr} = 2u_r + r u_{rr}$$

$$\Rightarrow v_{tt} = c^2 v_{rr}$$

$$v(0,r) = r\phi(r)$$

$$v_t(0,r) = r\psi(r)$$

$$\Rightarrow v(r,t) = \frac{(r-ct)}{2}\phi(r-ct) + \frac{(r+ct)}{2}\phi(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)\,ds$$

$$\Rightarrow u(r,t) = \frac{(r-ct)\phi(r-ct)}{2r} + \frac{(r+ct)\phi(r+ct)}{2r} + \frac{1}{2rc}\int_{r-ct}^{r+ct} s\psi(s)\,ds$$

■

#1

Solution:

(a) $V_x = -RI - LI_t$

$$I_x = -GV - CV_t$$

$$\Rightarrow V_{xx} = -RI_x - LI_{tx}$$

$$= RGV + RCV_t + LGV_t + LCV_{tt}$$

$$= LCV_{tt} + (RC+LG)V_t + RGV.$$

(c) The telegrapher's equation is

$$V_{tt} + aV_t + bV = c^2 V_{xx}$$

Letting $V(t,x) = e^{-\lambda t}u(t,x)$ we have that

$$V_t = -\lambda e^{-\lambda t}u + e^{-\lambda t}u_t$$

$$V_{tt} = \lambda^2 e^{-\lambda t}u - 2\lambda e^{-\lambda t}u_t + e^{-\lambda t}u_{tt}$$

$$V_{xx} = e^{-\lambda t}u_{xx}$$

$$\Rightarrow \lambda^2 u - 2\lambda u_t + u_{tt} - a\lambda u + au_t + bu = c^2 u_{xx}$$

If $\lambda = a/2$ we obtain

$$u_{tt} + \left(\frac{a^2}{4} - \frac{a^2}{2} + b\right)u = c^2 u_{xx}$$

Therefore,

$$K = b - \frac{a^2}{4}$$

(d) $K = c^2 RG - \frac{c^4}{4}(LG+RC)^2$

Letting $c^2 = 1/LC$ and $RC = LG$ we obtain

$$K = \frac{RG}{LC} - \frac{1}{L^2C^2}\,\frac{4L^2G^2}{4}$$

$$= \frac{RG}{LC} - \frac{G^2}{C^2}$$

$$= \frac{RG}{LC} - \frac{R}{L}\cdot\frac{G}{C}$$

$$= 0$$

#2

Solution:

(a) Let $E(t) = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right)dx$. Therefore,

$$\frac{dE}{dt} = \int_{-\infty}^{\infty}\left(u_t \cdot u_{tt} + c^2 u_x u_{xt}\right)dx$$

$$= \int_{-\infty}^{\infty}\left(u_t(c^2u_{xx} - au_t) + c^2u_xu_{xt}\right)dx$$

$$= \int_{-\infty}^{\infty}\left(-au_t^2 + c^2\cancel{u_{xx}u_t} - c^2\cancel{u_{xx}u_t}\right)dx - \cancel{c^2u_xu_t\Big|_{-\infty}^{\infty}}^{\,0}$$

$$= \int_{-\infty}^{\infty} -au_t^2\,dx.$$

Therefore, $\frac{dE}{dt} \leq 0$.

(b). Assume u_1, u_2 solve the telegrapher's equation and let $v = u_2 - u_1$.

Therefore, v satisfies the equation

$$v_{tt} + av_t = c^2 v_{xx}$$

$$v(0,x) = 0$$

$$v_t(0,x) = 0.$$

Consequently, the energy associated with v is given by

$$E(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2} v_t^2 + \frac{c^2}{2} v_x^2 \right) dx$$

and thus satisfies $E(0)=0$. Since $E \geq 0$, $\frac{dE}{dt} \leq 0$ and $E(0)=0$ it follows that $E=0$ for all t. Therefore, $v_t = v_x = 0 \Rightarrow v=0$. ∎