MTH 352/652: Homework #4

Due Date: February 23, 2024

1 Problems for Everyone

1. pg. 72, #3.12, #3.14, #3.15, #3.16.
2. Consider the following initial boundary value problem on the domain $[0, \pi]$:
$$\begin{cases} tu_t = u_{xx} + 2u \\ u(t,0) = u(t,\pi) \\ u(0,x) = 0 \end{cases}.$$
By separating variables, show that this initial boundary value problem has an infinite number of solutions.
3. By considering the energy
$$E(t) = \int_{-\pi}^{\pi} u^2 dx,$$
prove that solutions to the heat equation
$$\begin{cases} u_t = u_{xx} \\ u(0,x) = f(x) \\ u(t,-\pi) = u(t,\pi) \\ u_x(t,-\pi) = u_x(t,\pi) \end{cases}$$
are unique. **Hint:** First prove that $E(t)$ is decreasing in time.
4. Let f be a periodic function of period p, i.e., for all $x \in \mathbb{R}$, $f(x+p) = f(x)$.
   (a) Prove that for any $a \in \mathbb{R}$:
   $$\int_0^p f(x)dx = \int_a^{a+p} f(x)dx.$$
   Hint: Write $\int_a^{a+p} f(x)dx$ as the sum of two integrals (a to p and p to $a+p$) and make an appropriate change of variables.
   (b) Prove that for any $a \in \mathbb{R}$:
   $$\int_0^p f(x+a)dx = \int_0^p f(x)dx.$$
   (c) Interpret these identities graphically.
5. pg. 76 #3.2.1, #3.2.2.

Homework #4

#3.1.2

Find all seperable solutions to the heat equation

$$u_t = u_{xx}$$

on the interval $0 \le x \le \pi$ subject to

(a) homogeneous Dirichlet boundary conditions $u(t,0) = u(t,\pi) = 0$.

(b) mixed boundary conditions $u(t,0) = 0$, $u_x(t,\pi) = 0$.

(c) Neumann boundary conditions $u_x(t,0) = u_x(t,\pi) = 0$.

Solution:

Letting $u(t,x) = T(t)X(x)$ we have

$$T'X = TX''$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = -w^2, \quad w \ge 0$$

$$\Rightarrow T_n = ce^{-w^2 t}$$

$$X_n = a_n \cos(wx) + b_n \sin(wx);$$

Let $u_n(t,x) = a_n e^{-w^2 t} \cos(wx) + b_n e^{-w^2 t} \sin(wx)$

(a) Applying boundary conditions we have that

$$u_n(t,0) = a_n \cos(wx) = 0$$

$$\Rightarrow a_n = 0.$$

$$u_n(t,\pi) = b_n \sin(w\pi) = 0$$

$$\Rightarrow w = n, \ n \in \mathbb{N}.$$

Therefore,

$$u_n(t,x) = b_n e^{-n^2 t} \sin(nx).$$

(b) It follows from (a) that $a_n = 0$ again. Now,

$$\left.\frac{\partial u_n}{\partial x}\right|_{x=\pi} = b_n w \cos(w\pi) = 0.$$

$$\Rightarrow w = \frac{(2n+1)}{2} \Rightarrow w = n + \tfrac{1}{2}, \ n \in \mathbb{Z}^+.$$

$$\Rightarrow u_n(t,x) = b_n e^{-(n+\frac{1}{2})^2 t} \sin((n+\tfrac{1}{2})x)$$

(c) Applying boundary conditions we have that

$$\frac{\partial u_n}{\partial x}\Big|_{x=0} = -w a_n \sin(wx) + w b_n \cos(wx)\Big|_{x=0} = w b_n = 0$$

$$\Rightarrow b_n = 0.$$

$$\frac{\partial u_n}{\partial x}\Big|_{x=\pi} = -w a_n \sin(wx)\Big|_{x=\pi} = 0$$

$$\Rightarrow w = n, \quad n \in \mathbb{Z}^+.$$

$$\Rightarrow u_n(t,x) = \begin{cases} a_0, & n=0 \\ a_n e^{-n^2 t}\cos(nx), & n \in \mathbb{N}. \end{cases}$$

$\blacksquare$

#3.14

Find all seperable solutions to the following partial differential equations.

(a) $u_t = u_x$

(b) $u_t = u_x - u$

(c) $u_t = x u_x$

Solution:

(a) Let $u(t,x) = T(t)X(x)$. We have that

$$T'X = TX'$$

$$\Rightarrow \frac{T'}{T} = \frac{X'}{X} = \lambda$$

$$\Rightarrow T = c_1 e^{\lambda t}, \quad X = c_2 e^{\lambda x}$$

$$\Rightarrow u(t,x) = c e^{\lambda(t+x)}.$$

(b) Let $u(t,x) = T(t)X(x)$. We have that

$$T'X = TX' - TX$$

$$\Rightarrow \frac{T'}{T} = \frac{X'}{X} - 1 = \lambda$$

$$\Rightarrow T = c_1 e^{\lambda t}, \quad X = c_2 e^{(1+\lambda)x}$$

$$\Rightarrow u(t,x) = c e^{\lambda t + (1+\lambda)x}.$$

(c) Let $u(t,x) = T(t)X(x)$. We have that

$$T'X = xTX'$$

$$\Rightarrow \frac{T'}{T} = \frac{xX'}{X} = \lambda$$

$$\Rightarrow T = c_1 e^{\lambda t}, \quad X = c_2 x^{\lambda}$$

$$\Rightarrow u(t,x) = c e^{\lambda t} x^{\lambda}.$$

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#3

By considering the energy

$$E(t) = \int_{-\pi}^{\pi} u^2 \, dx$$

prove that solutions to the heat equation

$$\begin{cases} u_t = u_{xx} \\ u(0,x) = f(x) \\ u(t,-\pi) = u(t,\pi) \\ u_x(t,-\pi) = u_x(t,\pi) \end{cases} \qquad (*)$$

are unique.

Solution:

Computing we have that

$$\begin{aligned} \frac{dE}{dt} &= \int_{-\pi}^{\pi} 2u \cdot u_t \, dx \\ &= \int_{-\pi}^{\pi} 2u \cdot u_{xx} \, dx \\ &= 2u u_x \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2u_x^2 \, dx \\ &= \underbrace{2\big(u(t,\pi)u_x(t,\pi) - u(t,-\pi)u_x(t,-\pi)\big)}_{0} - \int_{-\pi}^{\pi} 2u_x^2 \, dx \\ &= \int_{-\pi}^{\pi} 2u_x^2 \, dx \\ &\leq 0. \end{aligned}$$

Suppose u_1, u_2 solve (*) there $v = u_2 - u_1$ solves

$$\begin{cases} v_t = v_{xx} \\ v(0,x) = 0 \\ v(t,-\pi) = v(t,\pi) \\ v_x(t,-\pi) = v_x(t,\pi) \end{cases}$$

Letting $E(t) = \int_{-\pi}^{\pi} v^2 dt$ it follows that

(1) $E(0) = 0$

(2) $\frac{dE}{dt} \leq 0$

(3) $E(t) \geq 0$

Properties (1)-(3) imply that $E(t) = 0$ for all time. Consequently,

$$v^2 = 0 \Rightarrow v = 0 \Rightarrow u_1 = u_2.$$

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#2.

Consider the following initial boundary value problem on the domain $[0,\pi]$:

$$\begin{cases} t u_t = u_{xx} + 2u \\ u(t,0) = u(t,\pi) \\ u(0,x) = 0 \end{cases}$$

By seperating variables, show that the initial boundary value problem has an infinite number of solutions.

Solution:

Let $u(t,x) = T(t)X(x)$. Therefore,

$$tT'X = TX'' + 2TX$$

$$\Rightarrow \frac{tT'}{T} = \frac{X''}{X} + 2 = \lambda$$

$$\Rightarrow \frac{tT'}{T} = \lambda, \quad X'' = (\lambda - 2)X$$

$$\Rightarrow T = ct^{\lambda}, \quad X(x) = \begin{cases} c_1 e^{\sqrt{\lambda-2}\,x} + c_2 e^{-\sqrt{\lambda-2}\,x}, & \lambda \geq 2 \\ c_1 \cos(\sqrt{2-\lambda}\,x) + c_2 \sin(\sqrt{2-\lambda}\,x), & \lambda < 2 \end{cases}$$

If we let $\lambda = 2$, we obtain

$$u(t,x) = ct^2$$

Which satisfies the PDE, initial conditions, and boundary conditions. Since c is arbitrary, solutions are not unique. ∎

#3.2.1

Find the Fourier series of the following functions

(a) $\text{sign}(x)$

(d) x^2

(e) $\sin^3(x)$

(h) $x\cos(x)$.

Solution:

Let $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$.

(a) $f(x) = \text{sign}(x)$. Note, $\text{sign}(x)$ is an odd function.

$$\Rightarrow a_0 = \langle \text{sign}(x), 1 \rangle = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_0^{\pi} 1\,dx = 0.$$

$$\Rightarrow a_n = \langle \text{sign}(x), \cos(nx) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}(x)\cos(nx)\,dx = 0.$$

$$\Rightarrow b_n = \langle \text{sign}(x), \sin(nx) \rangle$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos(nx)\,dx$$

$$= \frac{2}{\pi n} \sin(nx)\Big|_0^{\pi}$$

$$= \frac{2}{\pi n}(-1)^n.$$

$$\Rightarrow \text{sign}(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n}\sin(nx).$$

(d) $f(x)=x^2$. Note, x^2 is an even function.

$$\Rightarrow a_0 = \langle 1, x^2\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}.$$

$$\begin{aligned}
\Rightarrow a_n &= \langle \cos(nx), x^2\rangle \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\,x^2\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} \cos(nx)\,x^2\,dx \\
&= \cancelto{0}{\frac{2}{n\pi}\sin(nx)\,x^2\Big|_0^{\pi}} - \frac{2}{n\pi}\int_0^{\pi}\sin(nx)\cdot 2x\,dx \\
&= -\frac{4}{n\pi}\int_0^{\pi} x\sin(nx)\,dx \\
&= -\frac{4}{n\pi}\left(-\frac{1}{n}x\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right) \\
&= \frac{4}{n^2}\left(\cos(n\pi)-1\right) \\
&= \frac{4}{n^2}\left((-1)^n-1\right) \\
&= \frac{-8}{(2n-1)^2},
\end{aligned}$$

$$\Rightarrow x^2 \sim \frac{2\pi^2}{3} - 8\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}.$$

(e) Since $\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$ it follows that

$$\sin^3(x) \sim \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x).$$

(h) $f(x) = x\cos(x)$. Note, $f(x)$ is an odd function.

$$\Rightarrow a_0 = a_n = 0$$

$$\begin{aligned}
\Rightarrow b_n &= \langle x\cos(x), \sin(nx)\rangle \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\left(\sin((1+n)x) - \sin((1-n)x)\right)dx \\
&= \frac{-2}{\pi(1+n)}x\cos((1+n)x)\Big|_0^{\pi} + \frac{2}{\pi(1+n)}\cancelto{0}{\int_0^{\pi}\cos((1+n)x)\,dx} \\
&\quad + \frac{2}{\pi(1-n)}x\cos((1-n)x)\Big|_0^{\pi} - \frac{2}{\pi(1-n)}\cancelto{0}{\int_0^{\pi}\cos((1-n)x)\,dx} \\
&= \begin{cases} 2, & n=1 \\ 2\left(\dfrac{(-1)^{n+1}}{(1-n)} - \dfrac{(-1)^{n+1}}{(1+n)}\right), & n\in\mathbb{N},\ n\geq 1 \end{cases}
\end{aligned}$$

$$\Rightarrow \begin{cases} 2, & n=1 \\ \dfrac{4n(-1)^n}{n^2-1}, & n \in \mathbb{N},\ n>1 \end{cases}$$

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#3.1.6

Find the real eigensolutions for the diffusive transport equation $u_t = u_{xx} - u$ subject to the boundary conditions $u(t,-\pi) = u(t,\pi)$ and $u_x(t,-\pi) = u_x(t,\pi)$.

Solution:

Let $u(t,x) = T(t)X(x)$. Therefore,

$$T'X = TX'' - TX$$

$$\rightarrow \frac{T'}{T} = \frac{X''}{X} - 1 = \lambda$$

Periodic boundary conditions imply $\lambda + 1 \leq 0$. Let $\omega^2 = -(\lambda+1)$ we have that

$$T = ce^{-(1+\omega^2)t}, \quad X = a\cos(\omega x) + b\sin(\omega x).$$

Boundary conditions imply $\omega = n$, $n \in \mathbb{Z}^+$. Therefore,

$$u_0(t,x) = a_0 e^{-t}$$

$$u_n(t,x) = a_n e^{-(1+n)t}\cos(nx) + b_n e^{-(1+n)t}\sin(nx)$$

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