Homework #6

#1

Show that

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} + \mathcal{O}(\Delta x^2)$$

Solution:

Expanding we have that

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 + \frac{1}{6}f'''(x)\Delta x^3 + \frac{1}{24}f^{iv}(x)\Delta x^4 + \mathcal{O}$$

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 - \frac{1}{6}f'''(x)\Delta x^3 + \frac{1}{24}f^{iv}(x)\Delta x^4 + \mathcal{O}(\Delta x^5)$$

$$\Rightarrow f(x+\Delta x) - 2f(x) + f(x-\Delta x) = (1+1-2)f(x) + (1-1)f'(x)\Delta x + \left(\frac{1}{2}+\frac{1}{2}\right)f''(x)\Delta x^2$$
$$+ \left(\frac{1}{6}-\frac{1}{6}\right)f'''(x)\Delta x^3 + \left(\frac{1}{24}+\frac{1}{24}\right)f^{iv}(x)\Delta x^4 + \mathcal{O}(\Delta x^5)$$
$$= f''(x)\Delta x^2 + \frac{1}{12}f^{iv}(x)\Delta x^4 + \mathcal{O}(\Delta x^5)$$

$$\Rightarrow \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} = f''(x) + \frac{1}{12}f^{iv}(x)\Delta x^2 + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

∎

#5.1.4

Construct finite difference approximations to the first and second derivatives of a function using its values at the points $x-k$, $x+h$, where $h, k \ll 1$.

Solution:

Expanding we have that

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \mathcal{O}(h^4)$$

$$f(x-k) = f(x) - f'(x)k + \frac{1}{2}f''(x)k^2 - \frac{1}{6}f'''(x)k^3 + \mathcal{O}(k^4)$$

Therefore,

$$f(x+h)-f(x-k)=(h+k)f'(x)+\tfrac{1}{2}f''(x)(h^2-k^2)+O(h^3)+O(k^3)$$

$$\Rightarrow \frac{f(x+h)-f(x-k)}{h+k}=f'(x)+\frac{1}{2}f''(x)(h-k)+\ldots$$

$$\Rightarrow f'(x)=\frac{f(x+h)-f(x-k)}{h+k}+O(h-k)$$

We also have that

$$kf(x+h)+hf(x-k)=(k+h)f(x)+\tfrac{1}{2}(kh^2+hk^2)f''(x)+\tfrac{1}{6}(kh^3-hk^3)f'''(x)+O(kh^4)+O(hk$$

$$\Rightarrow kf(x+h)-(k+h)f(x)+hf(x-k)=\tfrac{1}{2}hk(h+k)f''(x)+O(kh(h-k)(h+k))$$

$$\Rightarrow f''(x)=\frac{2f(x+h)}{h(h+k)}-\frac{2f(x)}{hk}+\frac{2f(x-k)}{k(h+k)}+O(h-k)$$

∎

5.1.5

(a) Construct a finite difference formula that approximates $u'(x)$ using the points x, $x+h$, $x+2h$.

(b) Find a finite difference formula for $u''(x)$ that involves the same three function values.

Solution:

(a) $$f(x+h)=f(x)+f'(x)h+\tfrac{1}{2}f''(x)h^2+\tfrac{1}{6}f'''(x)h^3+O(h^4)$$

$$f(x+2h)=f(x)+f'(x)2h+2f''(x)h^2+\tfrac{8}{6}f'''(x)h^3+O(h^4)$$

$$\Rightarrow -4f(x+h)+f(x+2h)=-3f(x)-2f'(x)h+\tfrac{2}{3}f'''(x)h^3+O(h^4)$$

$$\Rightarrow f'(x)=\frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}+O(h^2)$$

(b) We also have that

$$f(x+2h)-2f(x+h)=-f(x)+f''(x)h^2+f'''(x)h^3+O(h^4)$$

$$\Rightarrow f''(x)=\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}+O(h)$$

#5.1.6

Using the function values $u(x)$, $u(x+h)$, $u(x+3h)$, find a finite difference approximation of $u'(x)$.

Solution:

Taylor expanding, we have that

$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(x)h^3 + O(h^4)$$

$$u(x+3h) = u(x) + u'(x)3h + \tfrac{1}{2}u''(x)9h^2 + \tfrac{27}{6}u'''(x)h^3 + O(h^4)$$

$$\Rightarrow u(x+3h) - 9u(x+h) = -8u(x) - 6u'(x)h + 3u'''(x)h^3 + O(h^4)$$

$$\Rightarrow u'(x) = \frac{-u(x+3h) + 9u(x+h) - 8u(x)}{6h} + O(h^2)$$

#5.1.7

Using the function values $u(x)$, $u(x+h)$, $u(x+3h)$, find a finite difference approximation of $u''(x)$.

Solution:

From the above calculations we have

$$u(x+3h) - 3u(x+h) = -2u(x) + 3u''(x)h^2 + 4u'''(x)h^3 + O(h^4)$$

$$\Rightarrow u''(x) = \frac{u(x+3h) - 3u(x+h) + 2u(x)}{3h^2} + O(h).$$

#5.1.8

Find the order of the five-point centered finite difference approximation.

$$u'(x) = \frac{-u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h)}{12h}.$$

Solution:

$$u(x+2h) = u(x) + u'(x)2h + \frac{1}{2}u''(x)4h^2 + \frac{1}{6}u'''(x)8h^3 + \frac{1}{24}u^{(iv)}(x)16h^4 + \frac{1}{5!}u^{(v)}(x)32h^5 + \frac{1}{6!}u^{(vi)}(x)64h^6 + O(h^7$$

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + \frac{1}{24}u^{(iv)}(x)h^4 + \frac{1}{5!}u^{(v)}(x)h^5 + \frac{1}{6!}u^{(vi)}(x)h^6 + O(h^7)$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + \frac{1}{24}u^{(iv)}(x)h^4 - \frac{1}{5!}u^{(v)}(x)h^5 + \frac{1}{6!}u^{(vi)}(x)h^6 + O(h^7)$$

$$u(x-2h) = u(x) - u'(x)2h + \frac{1}{2}u''(x)4h^2 - \frac{1}{6}u'''(x)8h^3 + \frac{1}{24}u^{(iv)}(x)16h^4 - \frac{1}{5!}u^{(v)}(x)32h^5 + \frac{1}{6!}u^{(vi)}(x)64h^6 + O(h^7)$$

$$\Rightarrow -u(x+2h) + 8u(x+h) + 8u(x-h) + u(x-2h) = (-1+8-8+1)u(x) + (-2+8+8-2)u'(x)h + (-2+4-4+2)u''(x)h^2$$

$$(-4/3 + 4/3 + 4/3 - 4/3)u'''(x)h^3 + (-2/3 + 1/3 - 1/3 + 2/3)u^{(iv)}(x)h^4$$

$$+ 1/5!(-32+8+8-32)u^{(v)}(x)h^5 + O(h^6)$$

$$= 12u'(x)h - 2/5\,u^{(v)}(x)h^5 + O(h^6)$$

$$\Rightarrow u'(x) = \frac{-u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h)}{12h} + O(h^4)$$