Homework #7

#1.

Consider the following initial boundary value problem

$$u_t = u_{xx}$$

$$u(0,x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x & 1/2 \le x \le 1 \end{cases}$$

$$u_x(t,0) = 0$$

$$u_x(t,1) = 0.$$

(a) Find the solution to this PDE.

(b) Calculate the equilibrium distribution, i.e., the $t \to \infty$ limit.

Solution:

(a). The generic form of the solution is given by

$$u(t,x) = a_0/2 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos(n\pi x)$$

Therefore,

$$\begin{aligned} a_0 &= 2\int_0^1 u(0,x) \\ &= 2 \cdot (\text{Area of triangle}) \\ &= 2 \cdot 1/4 \\ &= 1/2 \end{aligned}$$

Furthermore,

$$\begin{aligned} a_n &= 2\int_0^1 u(0,x)\cos(n\pi x)\,dx \\ &= 2\left(\int_0^{1/2} x\cos(n\pi x)\,dx + \int_{1/2}^1 (1-x)\cos(n\pi x)\,dx\right) \\ &= 2\left(\cancel{\frac{x}{n\pi}\sin(n\pi x)\Big|_0^{1/2}} + \cancel{\frac{(1-x)}{n\pi}\sin(n\pi x)\Big|_{1/2}^1} - \int_0^{1/2}\frac{1}{n\pi}\sin(n\pi x)\,dx \right. \\ &\quad \left. + \frac{1}{n\pi}\int_{1/2}^1 \sin(n\pi x)\,dx\right) \\ &= 2\left(\frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_0^{1/2} - \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_{1/2}^1\right) \\ &= \frac{2}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n + \cos\left(\frac{n\pi}{2}\right)\right) \end{aligned}$$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) - 1 + (-1)^{n+1}\right) e^{-n^2\pi^3 t}\cos(n\pi x).$$

(b) Calculating we have that

$$\lim_{t\to\infty} u(t,x) = \frac{1}{4}.$$

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#2.

Consider the following initial boundary value problem

$$u_t = u_{xx}$$

$$u(0,x) = \begin{cases} x, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$u(t,-2)=0$$

$$u(t,2)=0$$

(a) Find the solution to this PDE using Fourier series.

(b) Calculate the equilibrium distribution, i.e., the $t\to\infty$ limit.

Solution:

Leting $y = x+2$, we obtain

$$u_t = u_{yy}$$

$$u(0,y) = \begin{cases} y-2, & |y-2|<1 \\ 0, & |y-2|>1 \end{cases}$$

$$u(t,0)=0$$

$$u(t,4)=0$$

Therefore, the generic form of the solution is given by

$$u(t,y) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/16} \sin\left(n\pi y/4\right).$$

$$\Rightarrow b_n = \frac{1}{2}\int_0^4 u(0,y)\sin(n\pi y/4)\,dy$$

$$= \frac{1}{2}\int_1^3 (y-2)\sin(n\pi y/4)\,dy$$

$$\Rightarrow b_n = \frac{-4}{2n\pi}(y-2)\cos\left(\frac{n\pi y}{4}\right)\Bigg|_1^3 + \frac{4}{2n\pi}\int_1^3 \cos\left(\frac{n\pi y}{4}\right)dy$$

$$= \frac{-2}{n\pi}\cos\left(\frac{3n\pi}{4}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi y}{4}\right)\Bigg|_1^3$$

$$= \frac{-2}{n\pi}\cos\left(\frac{3n\pi}{4}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{8}{n^2\pi^2}\sin\left(\frac{3n\pi}{4}\right) - \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right)$$

Therefore,

$$u(t,x) = \sum_{n=1}^{\infty}\left[\frac{-2}{n\pi}\left(\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{4}\right)\right) + \frac{8}{n^2\pi^2}\left(\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right)\right]$$
$$\times e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right).$$

Computing, we have that

$$\lim_{t\to\infty} u(t,x) = 0$$

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pg. 139, #4.1.9

Solve the heat equation when the right hand side of the bar of unit length is held fixed at temperature α while the left hand side is insulated.

Solution:

The equation is given by

$$u_t = \gamma u_{xx}$$
$$u_x(t,0) = 0$$
$$u(t,1) = \alpha$$
$$u(0,x) = f(x)$$

The steady state solution is $u^*(x) = \alpha$. Therefore, if we let $v = u - u^*$ we obtain the following equation:

$$V_t = \gamma V_{xx}$$
$$V_x(t,0) = 0$$
$$V(t,1) = 0$$
$$V(0,x) = f(x) - \alpha.$$

Therefore, seperating variables as $v(t,x) = X \cdot T$ we have that
$$X T' = \gamma X'' T$$
$$\Rightarrow \frac{1}{\gamma}\frac{T'}{T} = \frac{X''}{X} = -w^2$$
$$\Rightarrow X = A\cos(wx) + B\sin(wx)$$
$$X'(0) = 0 \Rightarrow B = 0$$
$$X(1) = 0 \Rightarrow A\cos(w) = 0$$
$$\Rightarrow w = \frac{(2n-1)\pi}{2}$$

Consequently,
$$v(t,x) = \sum_{n=1}^{\infty} b_n e^{-(2n-1)^2\pi^2/4\,\gamma t}\cos\left(\frac{(2n-1)\pi}{2}x\right),$$

where
$$b_n = 2\int_0^1 (f(x) - \alpha)\cos\left(\frac{(2n-1)\pi}{2}x\right)dx$$

$$\Rightarrow v(t,x) = \alpha + \sum_{n=1}^{\infty} b_n e^{-(2n-1)^2\pi^2/4\,\gamma t}\cos\left(\frac{(2n-1)\pi}{2}x\right).$$

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pg. 139, #4.1.12

Show that the time derivative $v = u_t$ of any solution to the heat equation is also a solution. If $u(t,x)$ satisfies $u(0,x) = f(x)$ what initial condition does $v(t,x)$ inherit?

Solution:

Differentiating, we have that

$$V_t = U_{tt} = U_{txx}$$
$$V_{xx} = U_{txx}$$
$$\Rightarrow V_t = V_{xx}.$$

Since $v(0,x) = U_t(0,x) = U_{xx}(0,x)$ it follows that

$$V(0,x) = f''(x).$$

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pg. 139, #4.1.13

Explain why the thermal energy $E(t) = \int_0^\ell u(t,x)dx$ is not constant for the Dirichlet initial boundary value problem on the interval $[0,\ell]$.

Solution:

- Differentiating, we have that

$$\frac{dE}{dt} = \int_0^\ell \frac{\partial u}{\partial t} dx$$
$$= \int_0^\ell \frac{\partial^2 u}{\partial x^2} dx$$
$$= u_x(t,\ell) - u_x(t,0)$$
$$\neq 0.$$

- Another argument is that since

$$u(t,x) = \sum b_n e^{-n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right)$$

It follows that

$$\int_0^l u(0,x)\,dx = \int_0^l f(x)\,dx$$

but

$$\lim_{t\to\infty} \int_0^l u(t,x)\,dx = 0$$

#4.

The cable equation with Dirichlet boundary conditions is given by

$$u_t = \gamma u_{xx} - \alpha u$$

$$u(0,x) = f(x)$$

$$u(t,0) = 0$$

$$u(t,1) = 0$$

where $\alpha, \gamma > 0$. By making the change of variables $u = e^{-\alpha t} v$, find the general solution to the cable equation with Dirichlet boundary conditions.

Solution:

Letting $v = e^{\alpha t} u$, we have that

$$v_t = \alpha e^{\alpha t} u + e^{\alpha t} u_t = \alpha e^{\alpha t} u + e^{\alpha t}(\gamma u_{xx} - \alpha u) = e^{\alpha t} \gamma u_{xx}$$

$$v_{xx} = e^{\alpha t} u_{xx}$$

$$\Rightarrow v_t = v_{xx}$$

$$v(0,x) = e^{\alpha \cdot 0} u(0,x) = f(x)$$

$$v(t,0) = e^{\alpha t} u(t,0) = 0$$

$$v(t,1) = e^{\alpha t} u(t,1) = 0$$

Therefore,

$$U(t,x) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 \gamma t} \sin(n\pi x),$$

where

$$b_n = 2\int_0^1 f(x) \sin(n\pi x)\,dx.$$

Consequently,

$$V(t,x) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 \gamma t} \sin(n\pi x)$$

#5.

The convection-diffusion equation is given by $u_t + cu_x = \gamma u_{xx}$ where $c, \gamma > 0$.

(a) Show that $v(t,x) = u(t, x+ct)$ solves the heat equation.

(b) What is the physical interpretation of the change of variables?

Solution:

(a) Differentiating we have that

$$v_t = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot c = u_t + cu_x = \gamma u_{xx}$$

$$v_{xx} = u_{xx}$$

Therefore,

$$v_t = \gamma v_{xx}.$$

(b) This change of variables corresponds to a change of reference frames.

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#6.

The lossy-convection-diffusion equation is given by

$$u_t = \gamma u_{xx} + cu_x - \alpha u$$

By making an appropriate change of variables, show that this equation can be transformed to the heat equation.

Solution:

Let $v(t,x) = e^{\alpha t} u(t, x-ct)$. Differentiating we have that

$$\begin{aligned} v_t &= \alpha e^{\alpha t} u(t,x-ct) + e^{\alpha t} u_t(t,x-ct) - e^{\alpha t} c u_x(t,x-ct) \\ &= \alpha e^{\alpha t} u(t,x-ct) + e^{\alpha t}\left(\gamma u_{xx}(t,x-ct) + cu_x(t,x-ct) - \alpha u(t,x-ct)\right) \\ &\quad - e^{\alpha t} c u_x(t,x-ct) \\ &= e^{\alpha t} \gamma u_{xx}(t,x-ct) = \gamma v_{xx} \end{aligned}$$

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#7

Write down the solution to the following intial-boundary value problems for the wave equation in the form of a Fourier series.

(a) $u_{tt}=u_{xx}$, $u(t,0)=u(t,\pi)=0$, $u(0,x)=1$, $u_t(0,x)=0$

(b) $u_{tt}=2u_{xx}$, $u(t,0)=u(t,\pi)=0$, $u(0,x)=0$, $u_t(0,x)=1$

(c) $u_{tt}=3u_{xx}$, $u(t,0)=u(t,\pi)=0$, $u(0,x)=\sin^3(x)$, $u_t(0,x)=0$.

(d) $u_{tt}=2u_{xx}$, $u_x(t,0)=u_x(t,\pi)=0$, $u(0,x)=-1$, $u_t(0,x)=1$

(e) $u_{tt}=u_{xx}$, $u_x(t,0)=u_x(t,1)=0$, $u(0,x)=x(1-x)$, $u_t(0,x)=0$

Solution:

(a) For Dirichlet boundary conditions and zero initial velocity, the generic solution is of the form

$$u(t,x)=\sum_{n=1}^{\infty} b_n \cos(nt)\sin(nx)$$

$$\Rightarrow 1=u(0,x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow \int_0^{\pi} \sin(nx)\,dx = b_n \int_0^{\pi} \sin^2(nx)\,dx = b_n \int_0^{\pi} \frac{1-\cos(2nx)}{2}\,dx = \frac{\pi b_n}{2}$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = \frac{-2}{n\pi}\left(\cos(n\pi)-\cos(0)\right) = \frac{-2}{n\pi}\left((-1)^n-1\right) = \frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$\Rightarrow u(t,x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \cos((2n-1)t)\sin((2n-1)x).$$

(b) Seperating variables yields

$$u(t,x)=\sum_{n=1}^{\infty} b_n \sin(\sqrt{2}nt)\sin(nx)$$

$$\Rightarrow U_t(0,x)=1$$
$$=\sum_{n=1}^{\infty} b_n\sqrt{2}n\sin(nx)$$

$$\Rightarrow b_n=\frac{2}{\pi\sqrt{2}n}\int_0^{\pi}\sin(nx)dx$$
$$=\frac{-\sqrt{2}}{\pi n^2}((-1)^n-1)$$
$$=\frac{\sqrt{2}}{\pi n^2}(1-(-1)^n)$$

$$\Rightarrow u(t,x)=\frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin(\sqrt{2}(2n-1)t)\cos((2n-1)x)$$

(c) Seperating variables yields

$$u(t,x)=\sum_{n=1}^{\infty} b_n\cos(\sqrt{3}nt)\sin(nx)$$

$$\Rightarrow u(0,x)=3/4\sin(x)-1/4\sin(3x)$$
$$=\sum_{n=1}^{\infty} b_n\sin(nx)$$

$$u(t,x)=3/4\cos(\sqrt{3}t)\sin(x)-1/4\cos(3\sqrt{3}t)\sin(3x).$$

(d) Seperating variables yields

$$u(t,x)=\frac{a}{2}+bt+\sum_{n=1}^{\infty}c_n\cos(\sqrt{2}nt)\cos(nx)+\sum_{n=1}^{\infty}d_n\sin(\sqrt{2}nt)\cos(nx)$$

Boundary conditions imply $a=-2$, $b=1$ and all other coefficients are 0.

$$\Rightarrow u(t,x)=1-t.$$

(e) Seperating variables yields the solution

$$u(t,x) = a_0/2 + \sum_{n\geq 1}^{\infty} a_n \cos(n\pi t)\cos(n\pi x).$$

$$\Rightarrow u(0,x) = x(1-x)$$
$$= a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$\Rightarrow a_n = 2\int_0^1 x(1-x)\cos(n\pi x)\,dx$$
$$= 2\left(\cancelto{0}{\frac{x(1-x)\sin(n\pi x)}{n\pi}}\Bigg|_0^1 - \int_0^1 \frac{(1-2x)\sin(n\pi x)}{n\pi}\,dx\right)$$
$$= 2\left(\frac{(1-2x)\cos(n\pi x)}{n^2\pi^2}\Bigg|_0^1 + 2\int_0^1 \cancelto{0}{\frac{\cos(n\pi x)}{n^2\pi^2}}\,dx\right)$$
$$= \frac{2}{n^2\pi^2}\left((-1)^{n+1}-1\right)$$

$$\Rightarrow a_0 = 2\int_0^1 x(1-x)\,dx$$
$$= 2\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1$$
$$= 2\left(1/2 - 1/3\right)$$
$$= 1/3$$

Therefore,

$$u(t,x) = \frac{1}{6} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \cos(2n\pi t)\cos(2n\pi x)$$

#8

Let $u(t,x)$ be the solution to the wave equation $u_{tt} = c^2 u_{xx}$ on the interval $0<x<L$ satisfying Dirichlet boundary conditions. The total energy of u at time t is

$$E(t) = \int_0^L \tfrac{1}{2}\left(u_t^2 + c^2 u_x^2\right)dx.$$

(a) Prove that is E is constant.

(b) Prove that the only solution with $u(0,x)=0$, $u_t(0,x)=0$ is the trivial solution.

(c) Prove that solutions are unique.

Solution:

(a) Differentiating we have that

$$\frac{dE}{dt} = \int_0^L (u_t u_{tt} + c^2 u_x u_{tx})\,dx$$
$$= \int_0^L (u_t \cdot c^2 u_{xx} + c^2 u_x u_{tx})\,dx$$
$$= c^2 u_t u_x \Big|_0^L + \int_0^L \cancel{(-u_{tx} c^2 u_x + c^2 u_x u_{tx})}^{0}\,dx$$
$$= c^2 u_t u_x \Big|_0^L$$

Since $u(t,0) = u(t,L) = 0$ it follows that $u_t(t,0) = u_t(t,L) = 0$.

Therefore,
$$\frac{dE}{dt} = 0.$$

(b) If $v(0,x) = v_t(0,x) = 0$ it follows that $E(0) = 0$.

Therefore, for all t, $E(t) = 0 \Rightarrow u_t^2 = u_x^2 = 0 \Rightarrow u = 0$.

(c) If we let u_1, u_2 solve the wave equation and $v = u_2 - u_1$ it follows that v solves the wave equation and $v(0,x) = v_t(0,x) = 0$. Therefore, $v = 0 \Rightarrow u_2 = u_1$. ∎