MTH 352/652: Homework #8

Due Date: April 12, 2024

1 Problems for Everyone

1. This problem shows that taking integrals of solutions can sometimes give you another solution. Consequently, integration is another method of superimposing, or adding, a *continuum of solutions.* **Hint:** The assumptions on c in these problems ensure that you switch the order of differentiation and integration.

   (a) Consider Laplace's equation $u_{xx} + u_{yy} = 0$ for $x \in \mathbb{R}$ and $y > 0$. Show that $e^{ky}\sin(kx)$ is a solution for any value of $k > 0$ and show that for any bounded and continuous function $c(k)$ that
   $$u(x,y) = \int_0^\infty c(k)e^{-ky}\sin(kx)dk$$
   is a solution to the same equation.

   (b) Consider the heat equation $u_t = u_{xx}$ for $x \in \mathbb{R}$ and $t > 0$. Show that $e^{-k^2t}\sin(kx)$ is a solution for any value of $k > 0$ and show that for any bounded and continuous function $c(k)$ that
   $$u(t,x) = \int_{-\infty}^\infty c(k)e^{-k^2t}\sin(kx)dk$$
   is also a solution.

2. In class we showed that linear, homogenous PDEs with constant coefficients admit complex solutions of the form
   $$u(t,x) = Ae^{i(kx-\omega(k)t)},$$
   which are called plane waves. The real and imaginary parts give real solutions:
   $$\text{Re}(u) = A\cos(kx - \omega(k)t) \text{ and } \text{Im}(u) = A\sin(kx - \omega(k)t).$$
   If $\omega(k)$ is complex the PDE is called dissipative or diffusive and if $\omega(k)$ is real and $\omega''(k) \neq 0$, the PDE is called dispersive. If the PDE is dispersive, it means that the phase velocity or wave speed $\omega(k)/k$ depends upon the wave number k. For the following PDEs, find the dispersion relationship, classify the PDE as dissipative, dispersive or neither, if it is dispersive find the phase velocity, and finally describe the qualitative behavior of plane wave solutions.

   (a) $u_t + u_x + u_{xxx} = 0$ (Shallow water wave equation)

   (b) $u_t = u_{xxxxx}$ (Super diffusion)

   (c) $u_t + u_x - u_{xxt} = 0$ (Another water wave equation)

   (d) $u_{tt} = u_{xx} - u_{xxxx}$. (Beam equation)

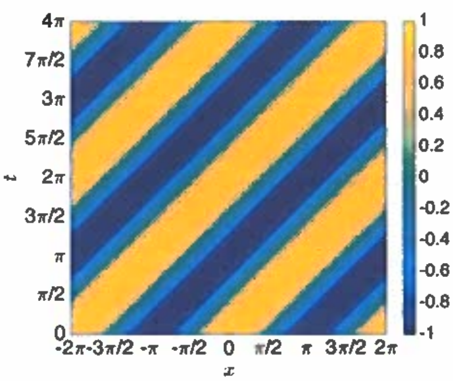
   (e) $u_t = -u - \delta u_{xx} - u_{xxxx}$, $\delta > 0$ (Flame front probagation equation)

3. Figures (a-c) below are contour plots for the *real component* of plane wave solutions $u(x,t) = e^{i(kx-\omega(k)t)}$ to an unknown linear partial differential equation.

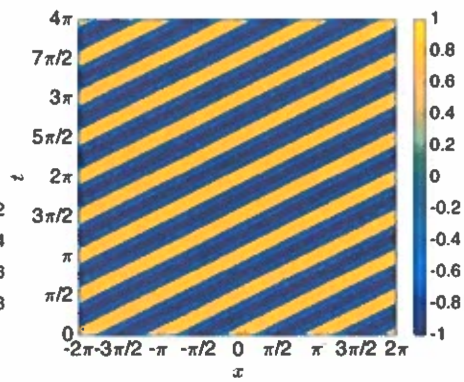
   (a) For each figure determine the value of k and ω. **Hint:** Think about how k and ω are related to spatial wavelength and temporal period.

   (b) Make a conjecture of the dispersion relationship $\omega(k)$.

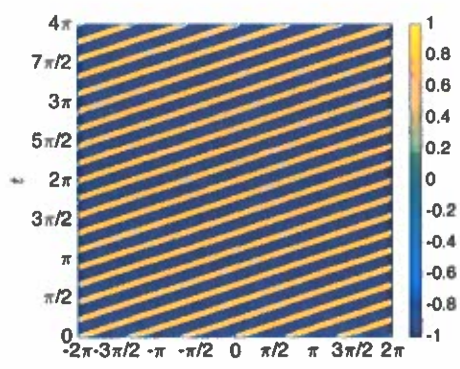
   (c) From this dispersion relationship determine the partial differential equation these plane wave solutions satisfy.



(a)



(b)



(c)

4. Show that Laplace's equation $u_{xx} + u_{yy} = 0$ is invariant under a translation of coordinates $\xi = x + a$, $\eta = y + b$; that is show

$$u_{\xi\xi} + u_{\eta\eta} = 0.$$

Show that solutions to Laplace's equation are also invariant under rotations by an angle α:

$$\begin{aligned} \xi &= x\cos(\alpha) + y\sin(\alpha), \\ \eta &= -x\sin(\alpha) + y\cos(\alpha). \end{aligned}$$

5. Recall that in polar coordinates $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$ that $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

(a) Show that

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}$$

(b) Show that

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\\ &= \cos^2(\theta)\frac{\partial^2}{\partial r^2} + 2\frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial \theta} - 2\frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\sin^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2}\\ \frac{\partial^2}{\partial y^2} &= \left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)\\ &= \sin^2(\theta)\frac{\partial^2}{\partial r^2} - 2\frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial \theta} + 2\frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\cos^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2}\end{aligned}$$

(c) Show that Laplace's equation $\Delta u = u_{xx} + u_{yy} = 0$ can be expressed in polar coordinates as

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

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7. Consider the following boundary value problem on a quarter wedge of radius R:

$$\begin{aligned}&\Delta u = 0, r < R,\ 0 \le \theta \le \pi/2,\\ &u(R,\theta) = \sin(2\theta),\\ &u(r,0) = u(r,\pi/2) = 0.\end{aligned}$$

(a) Solve this boundary value problem.

(b) Sketch a contour plot of your solution. (If you want to, you can use software to do this.)

8. Consider the following boundary value problem on an annulus:

$$\begin{aligned}&\Delta u = 0,\ 1 < r < 2,\ 0 \le \theta \le 2\pi,\\ &u(1,\theta) = 0,\\ &u(2,\theta) = \sin^2(\theta).\end{aligned}$$

(a) What additional boundary conditions must be imposed to make this problem well posed?

(b) Solve this boundary value problem.

(c) Sketch a contour plot of your solution. (If you want to, you can use software to do this.)

Homework #8

#2.

For the following PDEs, find the dispersion relationship, classify the PDE as dissipative, dispersive or neither, if it is dispersive find the phase velocity, and finally describle the qualitative behavior of plane wave solutions.

(a) $u_t + u_x + u_{xxx} = 0$

(b) $u_t = u_{xxxxx}$

(c) $u_t + u_x - u_{xxt} = 0$

(d) $u_{tt} = u_{xx} - u_{xxxx}$

(e) $u_t = -u - 8u_{xx} - u_{xxxx}$

Solution:

(a) If $u = Ae^{i(kx - w(k)t)}$ we have

$$-iw + ik - ik^3 = 0$$

$$\Rightarrow w = k - k^3$$

Therefore, this PDE is dispersive with phase velocity

$$V_p = 1 - k^2.$$

Consequently, plane wave travel to the right if $k^2 < 1$ are stationary if $k^2 = 1$ and travel to the left if $k^2 > 1$.

(b) If $u = Ae^{i(kx - w(k)t)}$ we have

$$-iw = ik^5$$

$$\Rightarrow w = -k^5$$

Therefore, this PDE is dispersvive with phase velocity

$$V_p = -k^4.$$

Consequently, plane wave travels to the left with speed k^4.

(c) In this case we have

$$-i\omega + ik - ik^2\omega = 0$$

$$\Rightarrow k = \omega(1+k^2)$$

$$\Rightarrow \omega = \frac{k}{1+k^2}$$

Consequently, this PDE is dispersive with phase velocity

$$V_p = \frac{1}{1+k^2}.$$

Therefore, wave travels to the right with velocity that vanishes for large k, i.e., short wavelength.

(d) In this case we have

$$-\omega^2 = -k^2 - k^4$$

$$\Rightarrow \omega = \pm k\sqrt{1+k^2}$$

Thus this is a dispersive PDE with phase velocity

$$V_p = \pm\sqrt{1+k^2}$$

Consequently, wave travels to the right and left at speed $\sqrt{1+k^2}$.

(e) In this case we have

$$-i\omega = -1 + \delta k^2 - k^4$$

$$\Rightarrow \omega = (-1+\delta k^2 - k^4)i$$

Therefore, this PDE is dissipative. The amplitude is given by

$$Ae^{(-1+\delta k^2 - k^4)t}$$

The amplitude grows in time if

$$g(k) = -1 + \delta k^2 - k^4 > 0$$

If we let $f(k^2) = -1 + \delta k^2 - (k^2)^2$ it follows that

$$f'(k^2) = \delta - 2k^2$$

Consequently, the local maximum of the quartic polynomial occur when $k^* = \pm\sqrt{\delta/2}$. Now,

$$g(k^*) = -1 + \frac{\delta^2}{2} - \frac{\delta^2}{4} = -1 + \frac{\delta^2}{4}$$

Therefore, there are growing amplitudes when

$$-1+\delta^2/4>0$$
$$\Rightarrow \delta^2>4$$
$$\Rightarrow \delta>2.$$

Consequently, we expect the fire to spread if $\delta>2$. ∎

#3.

(a) For each figure determine the value of k and ω.
(b) Make a conjecture for the dispersion relationship.
(c) From the dispersion relationship, determine the PDE these plane wave solutions satisfy.

Solution:

(a) Figure (a) we have

$$k=1$$
$$\omega=1$$

Figure (b) we have

$$k=2$$
$$\omega=4$$

Figure (c) we have

$$k=3$$
$$\omega=9$$

(b) $\omega=k^2$

(c) Schrodinger's equation works:

$$u_t=iu_{xx}.$$

∎

#4

Show that Laplace's equation is invariant under a translation of coordinates $\xi = x + a$, $\eta = y + b$. Show that solutions to Laplace's equation are also invariant under rotations by an angle α:

$$\xi = x\cos(\alpha) + y\sin(\alpha)$$
$$\eta = -x\sin(\alpha) + y\cos(\alpha)$$

Solution:

(a) By chain rule we have that

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}$$
$$\frac{\partial}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta}$$
$$\Rightarrow 0 = U_{xx} + U_{yy} = U_{\xi\xi} + U_{\eta\eta}$$

(b) By chain rule we have that

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \cos(\alpha)\frac{\partial}{\partial \xi} - \sin(\alpha)\frac{\partial}{\partial \eta}$$
$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \sin(\alpha)\frac{\partial}{\partial \xi} + \cos(\alpha)\frac{\partial}{\partial \eta}$$
$$\Rightarrow \frac{\partial^2}{\partial x^2} = \left(\cos(\alpha)\frac{\partial}{\partial \xi} - \sin(\alpha)\frac{\partial}{\partial \eta}\right)\left(\cos(\alpha)\frac{\partial}{\partial \xi} - \sin(\alpha)\frac{\partial}{\partial \eta}\right) = \cos^2(\alpha)\frac{\partial^2}{\partial \xi^2} - 2\cos(\alpha)\sin(\alpha)\frac{\partial^2}{\partial \xi \partial \eta} + \sin^2(\alpha)\frac{\partial^2}{\partial \eta^2}$$
$$\frac{\partial^2}{\partial y^2} = \left(\sin(\alpha)\frac{\partial}{\partial \xi} + \cos(\alpha)\frac{\partial}{\partial \eta}\right)\left(\sin(\alpha)\frac{\partial}{\partial \xi} + \cos(\alpha)\frac{\partial}{\partial \eta}\right) = \sin^2(\alpha)\frac{\partial^2}{\partial \xi^2} + 2\cos(\alpha)\sin(\alpha)\frac{\partial^2}{\partial \xi \partial \eta} + \cos^2(\alpha)\frac{\partial^2}{\partial \eta^2}$$

Therefore,

$$0 = U_{xx} + U_{yy} = (\cos^2(\alpha) + \sin^2(\alpha))U_{\xi\xi} + (\sin^2(\alpha) + \cos^2(\alpha))U_{\eta\eta} = U_{\xi\xi} + U_{\eta\eta}$$

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Solve the following boundary value problems

(a) $\Delta u = 0$, $x^2+y^2<1$, $u=x^3$, $x^2+y^2=1$

(b) $\Delta u = 0$, $x^2+y^2<2$, $u=\ln(x^2+y^2)$, $x^2+y^2=1$

(c) $\Delta u = 0$, $x^2+y^2<4$, $u=x^4$, $x^2+y^2<1$

(d) $\Delta u = 0$, $x^2+y^2<1$, $\frac{\partial u}{\partial n}=x$, $x^2+y^2=1$.

Solution:

(a) The generic solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

Boundary conditions imply

$$\cos^3\theta = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

$$\Rightarrow \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

$$\Rightarrow u(r,\theta) = \frac{3}{4} r\cos(\theta) + \frac{r^3}{4}\cos(3\theta).$$

(b) The generic solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

$$u(1,\theta) = 0 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$$

$$\Rightarrow u(r,\theta) = 0.$$

(c) The generic solution is

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n r^n\cos(n\theta)+\sum_{n=1}^{\infty}b_n r^n\sin(n\theta)$$

Boundary conditions imply that

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n 4^n\cos(n\theta)+\sum_{n=1}^{\infty}b_n 4^n\sin(n\theta)=4^4\cos^4\theta$$

$$=4^4\cdot\frac{1+\cos(2\theta)}{2}\cdot\frac{1+\cos(2\theta)}{2}$$

$$=4^3\left(1+2\cos(2\theta)+\cos^2(2\theta)\right)$$

$$=4^3\left(1+2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}\right)$$

$$=4^3\cdot\frac{3}{2}+4^3\cdot 2\cos(2\theta)+\frac{4^3}{2}\cos(4\theta)$$

Therefore,

- $\frac{a_0}{2}=\frac{4^3\cdot 3}{2}\Rightarrow a_0=192$
- $a_2\cdot 4^2=4^3\cdot 2\Rightarrow a_2=8$
- $a_4\cdot 4^4=\frac{4^3}{2}\Rightarrow a_4=\frac{1}{8}$.

Consequently,

$$u(r,\theta)=96+8\cdot r^2\cos(2\theta)+\frac{1}{8}r^4\cos(4\theta).$$

#8

Consider the following boundary value problem on an annulus:

$$\Delta u = 0, \quad 1 < r < 2, \quad 0 \le \theta \le 2\pi$$
$$u(1,\theta) = 0$$
$$u(2,\theta) = \sin^2\theta$$

Solve this boundary value problem.

Solution:

The generic solution is of the form

$$u(r,\theta) = a\ln(r) + b + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta)$$
$$+ \sum_{n=1}^{\infty} c_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} d_n r^{-n} \sin(n\theta)$$

Boundary conditions at $r=2$ imply:

$$\frac{1-\cos(2\theta)}{2} = a\ln(2) + b + \sum_{n=1}^{\infty} (a_n 2^n + c_n 2^{-n}) \cos(n\theta) + \sum_{n=1}^{\infty} (b_n 2^n + d_n 2^{-n}) \sin($$

$$\Rightarrow a\ln(2) + b = \frac{1}{2}, \quad 4a_2 + \frac{c_2}{4} = -\frac{1}{2}$$

and all other coefficients are zero. Boundary condition at $r=1$ implies

$$0 = b, \quad a_2 + c_2 = 0 \Rightarrow a_2 = -c_2.$$

Therefore,

$$a = \frac{1}{2\ln(2)}, \quad -4c_2 + \frac{c_2}{4} = -\frac{1}{2} \Rightarrow -15c_2 = -2 \Rightarrow c_2 = \frac{2}{15}$$

Consequently,

$$u(r,\theta) = \frac{\ln(r)}{2\ln(2)} - \frac{2}{15} r^2 \cos(2\theta) + \frac{2}{15} r^{-2} \cos(2\theta).$$

∎