Lecture 1: What are Partial Differential Equations?

Ordinary Differential Equations

Theorem - Consider the initial value problem

$$\frac{dx}{dt} = f(x,t) \qquad (1)$$

$$x(0) = x_0$$

If f is diffentiable and f' is continuous, then (1) has a a unique solution on some interval $(-\tau, \tau)$.

Example:

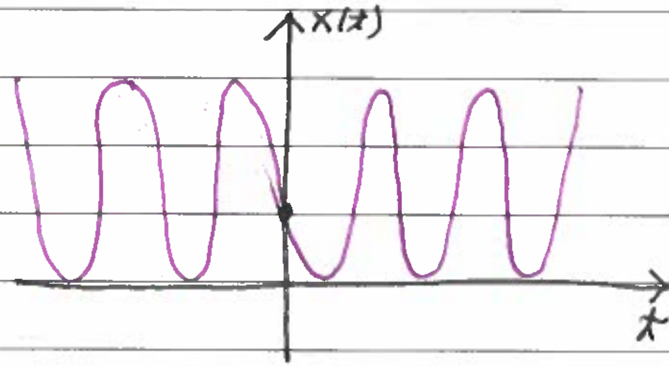
$$\frac{dx}{dt} = -\cos(t)x, \quad x(0) = x_0$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{x}\,dx = \int_0^t -\cos(t)\,dt$$

$$\Rightarrow \ln\left(\frac{x}{x_0}\right) = -\sin(t)$$

$$\Rightarrow x(t) = x_0 e^{-\sin(t)}.$$

(solution exists for all time).



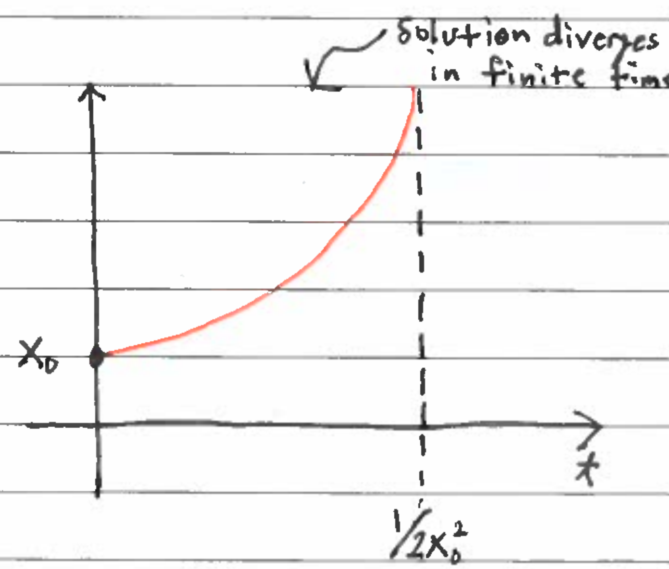
Example:

$$\frac{dx}{dt} = x^3, \quad x(0) = x_0$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{x^3}\,dx = \int_0^t dt$$

$$\Rightarrow -\frac{1}{2x^2} + \frac{1}{2x_0^2} = t$$

$$\Rightarrow x(t) = \frac{1}{\sqrt{x_0^{-2} - 2t}}$$



Why Care?

The theory of differential equations is essentially completely resolved.

(i) It is easy to predict when solutions are unique.

(ii) Solutions can blow up in finite time. Again this is easy to predict.

(iii) Solutions can run backwards in time.

(iv) There exist robust numerical solvers of ordinary differential equations.

* For partial differential equations there is no such robust theory *

Partial Differential Equations

A differential equation is called a partial differential equation if the function depends on more than one variable.

Example:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{(Leibniz notation)}$$

*u is a function of time t and space x.

$\Rightarrow u_t = u_{xx}$ (Subscript notation)

*The order of a PDE is the highest order derivative in the equation.

This is an example of a second order linear PDE.

Potential Solutions

$$u(t,x) = t + \tfrac{1}{4}x^2$$

$$u(t,x) = e^{-t}\sin(x)$$

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\exp\left(\frac{-x^2}{4t}\right)$$

How can we systematically find solutions?