Lecture 10: Stationary Solutions and Wave Equation

Example:

What is the stationary solution to

$$\bigstar \begin{cases} u_t = u_{xx} \\ u(t,0) = \alpha & \leftarrow \text{constant temperature} \\ u_x(t,\ell) = \beta & \leftarrow \text{constant flux} \\ u(0,x) = f(x) \end{cases}$$

Stationary solution $u^*(x)$ satisfies

$$u^*_{xx} = 0$$
$$u^*(0) = \alpha$$
$$u^*_x(\ell) = \beta$$
$$\Rightarrow u^*(x) = ax + b$$
$$u^*(0) = b = \alpha$$
$$u^*_x(0) = a = \beta$$

The stationary solution is given by

$$u^*(x) = \beta x + \alpha$$

To solve $\bigstar$ assume a solution of the form

$$u(t,x) = v(t,x) + u^*(x)$$
$$\Rightarrow v_t + u^*_t = v_{xx} + u^*_{xx}$$
$$u(t,0) = v(t,0) + \alpha = \alpha$$
$$u_x(t,0) = v_x(t,0) + \beta = \beta$$
$$u(0,x) = v(0,x) + ax + b = f(x)$$

Therefore, we obtain the following system

★★ $$\begin{cases} V_t = V_{xx} \\ V(t,0) = 0 \\ V_x(t,\ell) = 0 \\ V(0,x) = f(x) - ax - b \end{cases}$$

To solve ★★ we assume a separable solution

$$u(t,x) = T X$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = -w^2$$

Boundary conditions require that

$$X = A\sin(wx)$$

$$\Rightarrow X'(\ell) = 0 \Rightarrow \cos(w\ell) = 0$$

$$\Rightarrow w\ell = \frac{(2n-1)\pi}{2}$$

$$\Rightarrow w = \frac{(2n-1)\pi}{2\ell}$$

The solution is therefore,

$$u(t,x) = \sum_{n\geq 1}^{\infty} b_n \exp\left(-\frac{(2n-1)\pi^2}{4\ell^2}\right) \sin\left(\frac{(2n-1)}{2\ell}\pi x\right)$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} (f(x) - \beta x - \alpha)\sin\left(\left(\frac{2n-1}{2\ell}\right)\pi x\right) dx$$

Example:

What is the stationary solution of

$u_t = u_{xx} + \sin(x)$

$u(t,0)=0$

$u(t,1)=0$

Stationary solution satisfies:

$0 = u^*_{xx} + \sin(x)$

$\Rightarrow u^*_{xx} = -\sin(x)$

$\Rightarrow u^* = \sin(x) + ax + b$

$u^*(0) = b = 0$

$u^*(1) = \sin(1) + a \Rightarrow a = -\sin(1).$

Therefore,

$u^*(x) = \sin(x) - \sin(1)x$

Example:

$u_{tt} = c^2 u_{xx}$ ← Wave equation.

$u(0,x) = f(x)$

$u_t(0,x) = 0$

$u(t,0) = 0$

$u(t,1) = 0$

$$f(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1 \end{cases}$$

Assume $u(t,x) = XT$

$$\Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -w^2$$

$\Rightarrow X = \sin(n\pi x)$

$T = b_n \sin(n\pi c t) + d_n \cos(n\pi c t)$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} (b_n \sin(n\pi c t) + d_n \cos(n\pi c t)) \sin(n\pi x)$$

Initial conditions imply $b_n = 0$ and

$$d_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$\Rightarrow u(t,x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} (-1)^{k+1} \frac{\cos((2n-1)\pi t)\sin((2n-1)\pi x)}{(2n-1)^2}$$

Example:

$u_{tt} = c^2 u_{xx}$

$u(0,x) = 0$

$u_t(0,x) = f(x)$

$u_x(t,0) = 0$

$u_x(t,1) = 0$

$$f(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1 \end{cases}$$

$u(t,x) = TX$

$$\Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\omega^2$$

General Solution:

$X = \cos(n\pi x)$ or $X = A$

$T = a_n \sin(cn\pi t) + b_n \cos(cn\pi t)$ or $T = a_0 + a_0' t$

Initial Conditions imply

$a_0 = 0,\ b_n = 0$

$$u_t(0,x) = f(x) = a_0' + \sum_{n=1}^{\infty} a_n c n\pi \cos(n\pi x)$$

$$\Rightarrow a_0' = \int_0^1 f(x)\,dx$$

$$a_n = \frac{2}{c n\pi}\int_0^1 f(x)\cos(n\pi x)\,dx$$