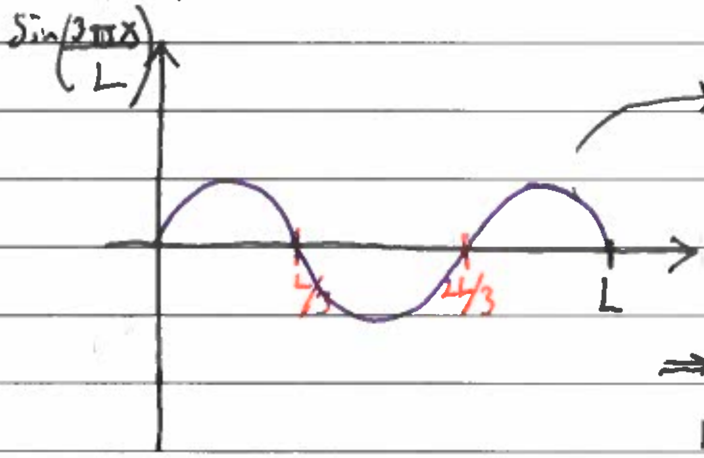
Lecture #11: Dispersion Relationships

Heat Equation Summary

$U_t = \gamma U_{xx}, \quad x \in [0, L]$

$$U = \sum_{n=1}^{\infty} b_n \underbrace{e^{-n^2\pi^2\gamma t/L^2}}_{\text{decaying amplitude}} \underbrace{\sin(n\pi x/L)}_{\text{stationary waves = modes}}$$ (Dirichlet Boundary Condition)

$$= \sum_{n=1}^{\infty} b_n U_n(t, x)$$



→ 3rd harmonic

k = wave number

$= \frac{n\pi}{L}$ (in this case $3\pi/L$)

⇒ k is a measurement of number of peaks in a spatially periodic wave.

* The third harmonic decays very quickly to zero since the amplitude is

$$b_3 e^{-81\pi^2\gamma t/L^2}$$

⇒ If b_n^* is the first nonzero term

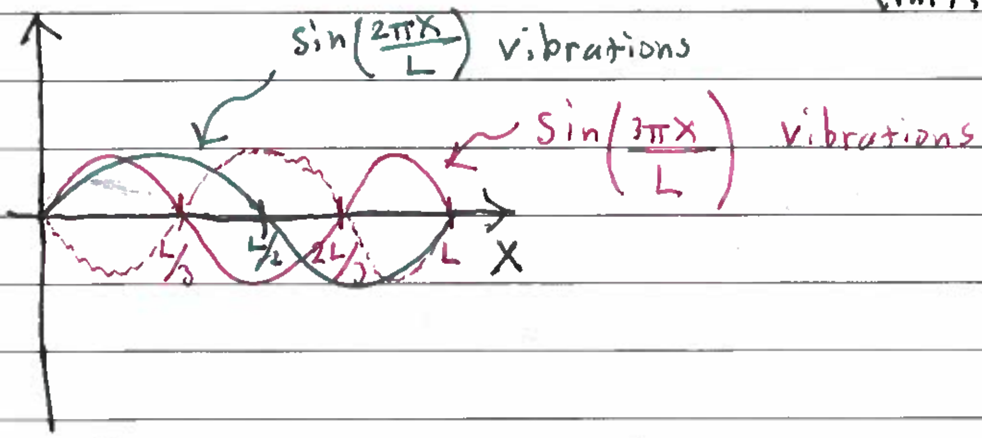
$$U(t, x) \approx \underbrace{b_n^* e^{-n^{*2}\pi^2\gamma t/L^2} \sin(n^*\pi x)}_{\text{forgets almost everything about initial condition.}}$$

* For $t > 0$, since $b_n e^{-n^2\pi^2\gamma t/L^2}$ decays exponentialy implies $u(t, x)$ is smooth for all $t > 0$!!!

Wave Equation Summary

$u_{tt} = c^2 u_{xx}, \quad x \in [0, L]$

$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi c t}{L}\right) \sin\left(\frac{n\pi x}{L}\right),$ (Dirichlet Boundary conditions and zero initial velocity)



* The harmonics are now periodic in time each with a period given by

$$T_n = \frac{2L}{nc} = \frac{2L}{c} \cdot \frac{1}{n}$$

The spatial period is given by

$$l_n = \frac{2L}{n} = 2L \cdot \frac{1}{n} = cT_n$$

$\Rightarrow \boxed{l_n = cT_n} \rightarrow$ Connection between space and time.

* At a later time, coefficients in Fourier series are given by $b_n \cos\left(\frac{n\pi c t}{L}\right)$

$\Rightarrow$ Solutions to wave equation are as smooth as initial data.

$\Rightarrow$ Information propagates at speed c.

Dispersion Relationships

From the wave-equation we learned about a coupling between

w = angular frequency

k = wave number

How do we find such relationships?

- Heat Equation:

$$u_t = \gamma u_{xx}$$

Assume $u(t,x) = e^{ikx - i\omega(k)t}$ (Plane Wave Solution)

$$\Rightarrow -i\omega(k)\, e^{ikx - i\omega(k)t} = \gamma (ik)^2 e^{ikx - i\omega(k)t}$$

$$\Rightarrow -i\omega(k) = -\gamma k^2$$

$$\Rightarrow \boxed{\omega(k) = -ik^2} \rightarrow \text{dispersion relationship}$$

$$\Downarrow$$

$$u(t,x) = \underbrace{e^{-k^2 t}\left(\cos(kx) + i\sin(kx)\right)}_{\text{Solution dissipates in time.}}$$

- Wave Equation:

$$u_{tt} = c^2 u_{xx}$$

Assume $u(t,x) = e^{ikx - i\omega(k)t}$

$$\Rightarrow -\omega(k)^2 = -c^2 k^2$$

$$\Rightarrow \boxed{\omega(k) = \pm ck} \rightarrow \text{dispersion relationship}$$

$$\Rightarrow u(t,x) = e^{ikx - ickt} \text{ or } e^{ikx + ickt}$$

$$\Downarrow$$

$$u(t,x) = \underbrace{\cos(k(x-ct)) + i\sin(k(x-ct))}_{\text{moves to the right at speed } c} \text{ or } \underbrace{\cos(k(x+ct)) + i\sin(k(x+ct))}_{\text{moves to the left at speed } c.}$$

Speed is independent of wave number k!

Schrondinger's Equation

$u_t = i u_{xx}$

Assume $u(t,x) = e^{i(kx - w(k)t)}$

$\Rightarrow -iw(k) = i(ik)^2$

$\Rightarrow \boxed{w(k) = k^2} \rightarrow$ dispersion relationship.

$\Rightarrow u(t,x) = e^{i(k \cdot x - k^2 t)}$

$\Rightarrow u(t,x) = \underbrace{\cos(k(x-kt)) + i\sin(k(x-kt))}_{\text{moves to the right at speed } k}$

$\Rightarrow$ wave disperses since different wave lengths move at different speeds.