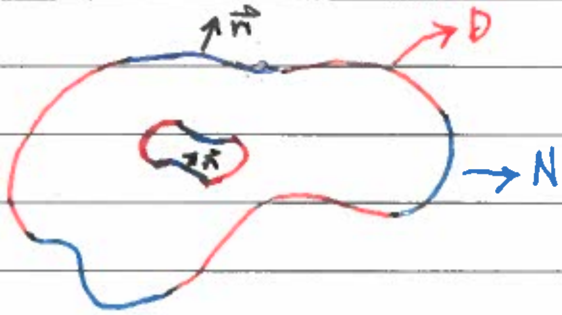
Lecture #12: Laplace's Equation



$u_t = u_{xx} + u_{yy}$

$u|_D = f(x,y)$ (Dirichlet)

$\nabla u \cdot \vec{n} = 0$ (Neumann)

$u(0,x,y) = g(x,y)$ (Initial Conditions

Steady State Distribution satisfies

$$u_{xx} + u_{yy} = \nabla \cdot \nabla u = \Delta u = 0.$$

Example:

$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le a,\ 0 \le y \le b\}.$

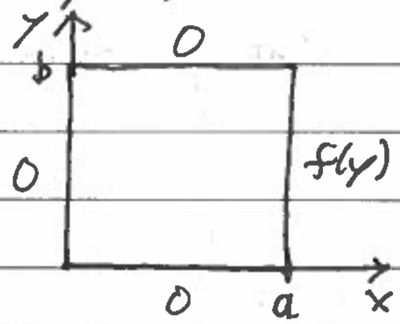
$\Delta u = 0$

$u(0,y) = 0$

$u(a,y) = f(y)$

$u(x,0) = 0$

$u(x,b) = 0$



$u(x,y) = X \cdot Y$

$\Rightarrow X''Y + XY'' = 0$

$$\Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} = -w^2$$

Boundary conditions imply

$$Y = b_n \sin\left(\frac{n\pi y}{b}\right), \quad w = \frac{n\pi}{b}.$$

$$X'' = w^2 X = \frac{n^2\pi^2}{b} X$$

$$\Rightarrow X = A\cosh(wx) + B\sinh(wx).$$

Boundary condition at $x=0$:

$$u(0,y)=0 \Rightarrow A\sin\left(\frac{n\pi y}{b}\right)=0 \Rightarrow A=0.$$

A generic solution is therefore of the form

$$u_n(x,y)=b_n\sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow u(x,y)=\sum_{n=1}^{\infty} b_n\sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

The last boundary condition implies that

$$u(a,y)=f(y)=\sum_{n=1} b_n\sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow \int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy=\frac{b_n}{2}\sinh\left(\frac{n\pi a}{b}\right)$$

$$\Rightarrow b_n=\frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy.$$

Example:

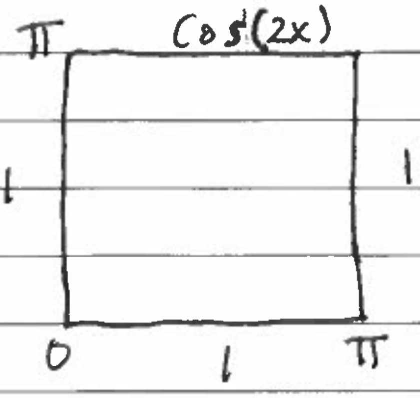
Solve $\Delta u=0$

$u(0,y)=1$

$u(\pi,y)=1$

$u(x,0)=1$

$u(x,\pi)=\cos(2x)$



$\tilde{U}(x,y)=1$ solves Laplace's equation. Define

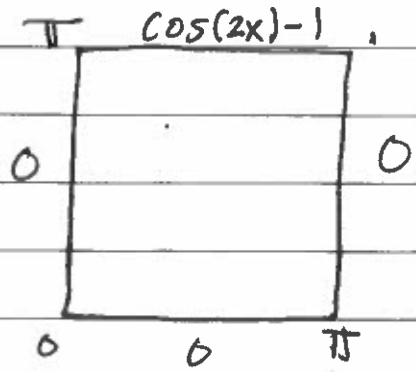
$$v(x,y)=u(x,y)-1$$

$$\Rightarrow \Delta v=0$$

$$\Rightarrow v(0,y)=v(x,0)=v(\pi,y)=1$$

$$\Rightarrow v(x,\pi)=\cos(2x)-1.$$

$$\begin{cases} \Delta v = 0 \\ v(0,y) = 0 \\ v(x,0) = 0 \\ v(\pi,y) = 0 \\ v(x,\pi) = \cos(2x) - 1 \end{cases}$$



Seperating variables yields

$$v(x,y) = \sum_{n=1}^{\infty} a_n \sin(nx)\sinh(ny)$$

$$\Rightarrow a_n = \frac{2}{\pi \sinh(n\pi)} \int_0^{\pi} \sin(nx)(\cos(2x) - 1)\,dx$$

$$\Rightarrow a_n = \frac{4(1-(-1)^n)}{n^3 - 4n} \cdot \frac{2}{\pi \sinh(n\pi)}$$

$$\Rightarrow u(t,x) = 1 + \sum_{\substack{n=1 \\ n \neq 2}}^{\infty} \frac{8(1-(-1)^n)}{\pi \sinh(n\pi)(n^3 - 4n)} \sin(nx)\sinh(ny).$$

Example:

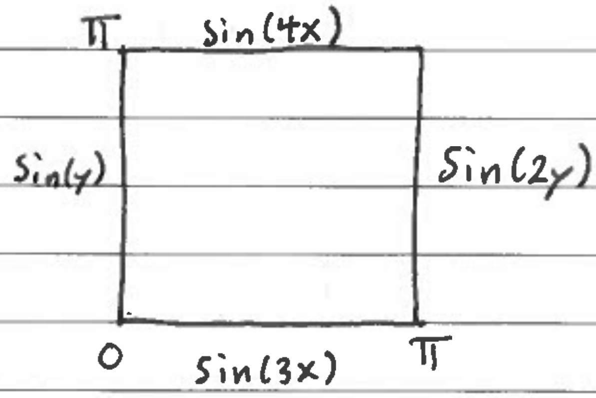
Solve $\Delta u = 0$

$u(0,y) = \sin(y)$

$u(\pi,y) = \sin(2y)$

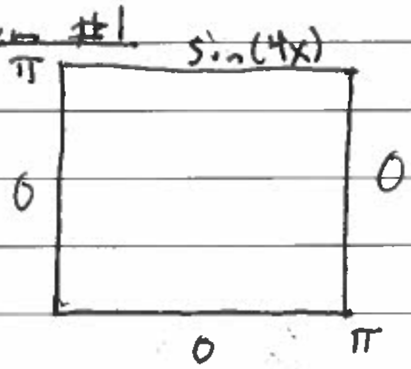
$u(x,0) = \sin(3x)$

$u(x,\pi) = \sin(4x)$



Idea: Split into four problems.

Problem #1



$$\Rightarrow u_n(x,y) = (A\cosh(ny) + B\sinh(ny))\cdot\sin(nx)$$

$$u_n(x,0) = 0 \Rightarrow A\cosh(y)\sin(nx) = 0$$

$$\Rightarrow A = 0$$

$$\rightarrow u_n(x,y) = B\sinh(ny)\sin(nx)$$

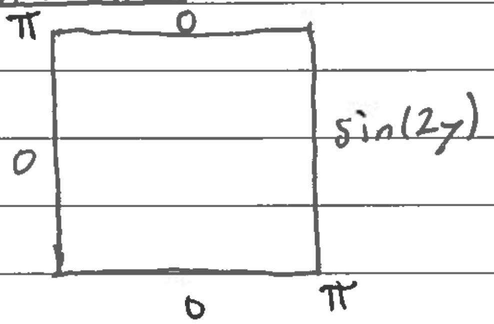
$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(ny)\sin(nx)$$

$$\Rightarrow u(x,\pi) = \sin(4x) = \sum_{n=1}^{\infty} b_n \sinh(ny)\sin(nx)$$

$$\Rightarrow b_4 = \frac{1}{\sinh(4\pi)}, \quad \text{all other } b_n = 0$$

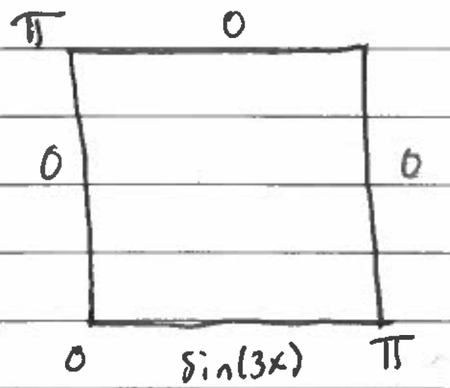
$$\Rightarrow u(x,y) = \frac{\sinh(4y)}{\sinh(4\pi)}\sin(4x).$$

Problem #2

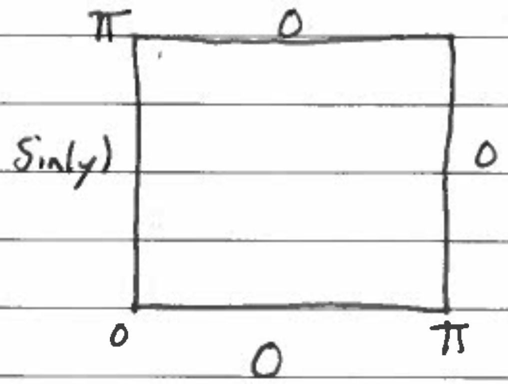


$$\Rightarrow u(x,y) = \frac{\sinh(2x)}{\sinh(2\pi)}\sin(2y)$$

Problem #3



Problem #4



$$u(x,y)=\frac{\sinh(3(\pi-y))}{\sinh(3\pi)}\sin(3x),\quad u(x,y)=\frac{\sinh(\pi-x)}{\sinh(\pi)}\sin(y)$$

Combing all of the solutions we have:

$$\boxed{u(x,y)=\frac{\sinh(4y)\sin(4x)}{\sinh(4\pi)}+\frac{\sinh(2x)\sin(2y)}{\sinh(2\pi)}+\frac{\sinh(3(\pi-y))\sin(3x)}{\sinh(3\pi)}+\frac{\sinh(\pi-x)\sin(y)}{\sinh(\pi)}}$$

Maximum Principle: If u solves Laplace's equation then u obtains it maximum and minimum on $\partial\Omega$.

~~proof~~ (2-D):

If u obtains max (min) in Ω then $\nabla u|_{(a,b)}=0$.

The discriment is

$$u_{xx}u_{yy}-u_{xy}^2=\det\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}=\lambda_1\cdot\lambda_2 \text{ (product of eigenvalu}$$

$$u_{xx}+u_{yy}=\operatorname{tr}\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}=\lambda_1+\lambda_2=0$$

$\Rightarrow \lambda_1=-\lambda_2$

$\Rightarrow$ discriment <0

$\Rightarrow$ saddle point (cannot be max or min).

(More technical proof needed, but this is the idea).

Theorem - Solutions to Laplace's equation with Dirichlet boundary conditions are unique.

proof:

Suppose u_1, u_2 solve Laplace's equation. Then $v = u_1 - u_2$ solves

$$\Delta v = 0$$

$$v|_{\partial\Omega} = 0$$

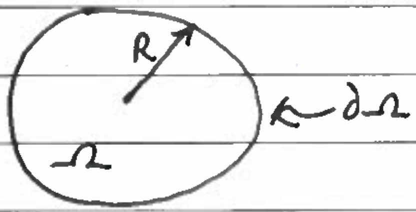
Therefore, $\max(v) = 0$, $\min(v) = 0 \Rightarrow v = 0$. ∎

Example:

$$0 = \Delta u$$

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}.$$

$$u(R,\theta) = f(\theta)$$



Convert to polar coordinates

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$u(R,\theta) = f(\theta)$$

$$\left.\begin{aligned} u(r,0) &= u(r,2\pi) \\ u_\theta(r,0) &= u_\theta(r,2\pi) \end{aligned}\right\} \text{periodic boundary conditions}$$

Assume

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Rightarrow \Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2}\Theta'' R = 0$$

$$\Rightarrow \frac{R'' + \frac{1}{r}R'}{R} r^2 = -\frac{\Theta''}{\Theta} = \omega^2$$

$$\Rightarrow \Theta = A\cos(\omega\theta) + B\sin(\omega\theta)$$

Periodic Boundary Conditions imply $\omega = n$.

We now solve for $R(r)$

$$r^2R''+rR'-w^2R=0$$

Guess $R=ar^m$

$$\Rightarrow am(m-1)r^m+amr^m-an^2r^m=0$$

$$\Rightarrow m^2-m+m-n^2=0$$

$$\Rightarrow m=\pm n$$

$$\Rightarrow R=ar^n+br^{-n}.$$

However at $r=0$, we need to set $b=0$ to avoid a singularity. The generic solution is

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n r^n\cos(n\theta)+\sum_{n=1}^{\infty}b_n r^n\sin(n\theta)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(\theta)d\theta$$

$$a_n=\frac{1}{\pi R^n}\int_0^{2\pi}f(\theta)\cos(n\theta)d\theta$$

$$b_n=\frac{1}{\pi R^n}\int_0^{2\pi}f(\theta)\sin(n\theta)d\theta$$

Example:

$$\Delta u=0$$

$$\Omega=\{(x,y)\in\mathbb{R}^2: x^2+y^2\leq R\}$$

$$u(R,\theta)=\sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\Rightarrow u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n r^n\cos(n\theta)+\sum_{n=1}^{\infty}b_n r^n\sin(n\theta)$$

$$\Rightarrow u(r,\theta)=\frac{1}{2}-\frac{r^2}{R^2}\cos(2\theta).$$