Lecture 14: Fourier Transforms

Return to Fourier Series:

On the domain $[\pi,\pi]$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

$$\Rightarrow f \longleftrightarrow \{a_n, b_n\}$$

Given $\{a_n, b_n\}$ a function can be constructed and given $f(x)$ a_n, b_n can be constructed via:

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \end{aligned}}$$

Converting to complex members we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) + \sum_{n=1}^{\infty} \frac{b_n}{2i}\left(e^{inx} - e^{-inx}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{inx} + \sum_{n=-\infty}^{-1} \frac{a_{-n}}{2} e^{inx} + \sum_{n=1}^{\infty} \frac{b_n}{2i} e^{inx} - \sum_{n=-\infty}^{-1} \frac{b_{-n}}{2i} e^{inx}$$

$$= \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \underbrace{\frac{a_n - i b_n}{2}}_{c_n\ (n>0)} e^{inx} + \sum_{n=-\infty}^{-1} \underbrace{\frac{a_{-n} + i b_{-n}}{2}}_{c_n\ (n<0)} e^{inx}$$

$$\Rightarrow \boxed{\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \end{aligned}}$$

Fourier series pair

Fourier Transforms:

The Fourier transform generalizes the idea of a Fourier series to functions on $\mathbb{R}$

- $\mathcal{F}[f](k) = \underset{\substack{\uparrow \\ \text{Amplitude} \\ \text{of k-th mode}}}{\hat{f}(k)} = \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx}_{\text{projection onto k-th mode.}}$ (Fourier transform)

- $\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)e^{ikx}dk$ (Inverse Fourier transform)

Examples:

1. $f(x) = e^{-ax^2}$, $(a>0)$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{-ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2-ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a(x^2+ik/a\, x)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(-a\left(x^2+\frac{ik}{a}x-\frac{k^2}{4a}+\frac{k^2}{4a^2}\right)\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(-a\left(x+\frac{ik}{a}\right)^2\right)e^{-k^2/4a}dx$$

$$= \frac{1}{\sqrt{2\pi}}e^{-k^2/4a}\int_{-\infty}^{\infty} \exp\left(-a\left(x+\frac{ik}{a}\right)^2\right)dx$$

Derivatives and Integrals:

1. $\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-ikx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (ik)f(x)e^{-ikx}dx$$

$$= (ik)\,\mathcal{F}[f]$$

Integrate by parts

2. $\mathcal{F}[f''(x)] = (ik)^2\mathcal{F}[f]$

$$= -k^2\mathcal{F}[f]$$

3. Let $F(x) = \int_{-\infty}^{x} f(s)ds$ (antiderivative).

$$\Rightarrow \mathcal{F}[F(x)](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)e^{-ikx}dx$$

$$= \frac{-1}{\sqrt{2\pi}}\frac{1}{(-ik)}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$= \frac{i}{k}\mathcal{F}[f](k).$$

Convolutions:

The convolution of $f(x)$ and $g(x)$ is defined by

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

Theorem:

If $h(x) = (f*g)(x)$ then

$$\hat{h} = \sqrt{2\pi}\,\hat{f}\cdot\hat{g}. \Rightarrow h = \sqrt{2\pi}\,\mathcal{F}^{-1}[\hat{f}\hat{g}] \Rightarrow f*g = \sqrt{2\pi}\,\mathcal{F}^{-1}[\hat{f}\hat{g}]$$

Also if $h(x) = f(x)g(x)$ then

$$\hat{h} = \frac{1}{\sqrt{2\pi}}\hat{f}*\hat{g}$$

$$\Rightarrow \boxed{\mathcal{F}[e^{-ax^2}] = \frac{1}{\sqrt{2a}} e^{-k^2/4a}}$$

2. $f(x) = e^{-|x|}$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{x} e^{-ikx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x} e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-ik} + \frac{1}{1+ik}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{1+k^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}$$

Properties:

1. $\mathcal{F}[af] = a\mathcal{F}[f]$.
2. $\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax) e^{-ikx}\,dx$
$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a}\int_{-\infty}^{\infty} f(u)\, e^{-iku/a}\,du$$
$$= \frac{1}{a}\ \mathcal{F}[f](k/a)$$
3. $\mathcal{F}[f](k) = \mathcal{F}^{-1}[f](-k)$.
4. $\mathcal{F}[f(x-a)](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a) e^{-ikx}\,dx$
$$= \frac{1}{\sqrt{2\pi}} e^{-ika}\int_{-\infty}^{\infty} f(u) e^{-iku}\,du$$
$$= e^{-ika}\,\mathcal{F}[f](k).$$

~~proof~~:

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x) e^{-ikx} dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y) g(y) e^{-ikx} dy\, dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y) g(y) e^{-ikx} dx\, dy$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u) g(y) e^{-iku} e^{-iky} du\, dy$$
$$= \sqrt{2\pi}\, \hat{f}(k)\hat{g}(k).$$

Example:

Solve:

$$u_t = u_{xx}, \quad x \in \mathbb{R},$$
$$u(0,x) = f(x)$$

Take Fourier transforms:

$$\hat{u}_t = -k^2 \hat{u}$$
$$\hat{u}(0,k) = \hat{f}(k)$$

Therefore,

$$\hat{u}(t,k) = \hat{f}(k) e^{-k^2 t}$$
$$\Rightarrow u(t,x) = \mathcal{F}^{-1}[\hat{f} e^{-k^2 t}]$$
$$= \frac{1}{\sqrt{2\pi}} f * \mathcal{F}^{-1}[e^{-k^2 t}]$$
$$= \frac{1}{\sqrt{2\pi}} f * \mathcal{F}[e^{-k^2 t}](-x)$$
$$= \frac{1}{\sqrt{4\pi t}} f * \exp\left(\frac{-x^2}{4t}\right)$$

$$\Rightarrow \boxed{u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)\, dy}$$

The Green's function or heat kernel is

$$\boxed{G(x,y) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right)}$$