Lecture 15: Distributions (Generalized Functions/Measures)

Solving Linear Equations:

Let $A \in \mathbb{R}^{n \times n}$. Solve

$$A\vec{x} = \vec{b}$$
$$\Rightarrow \vec{x} = A^{-1}\vec{b}$$

The solution exists and is unique if A is nonsingular.

Example:

Find the inverse of

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

Row reduce:

$$A^{-1} = \begin{bmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{bmatrix}$$

Why does this work???

Inverse of a matrix (abstract)

How does a matrix work. Let $\vec{x} \in \mathbb{R}^n$ such that

$$\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n, \quad \vec{e}_i = [0, 0, \dots, 0, \underset{\text{i-th slot}}{1}, 0, \dots, 0]^T$$

$$A\vec{x} = x_1 A\vec{e}_1 + \dots + x_n A\vec{e}_n,$$
$$= x_1\vec{c}_1 + \dots + x_n\vec{c}_n,$$

$\vec{c}_1, \dots, \vec{c}_n$ are the columns of A.

$\Rightarrow$ To find columns of A need to compute $\vec{e}_i$:

To find the columns of A^{-1} need to figure out

$$A^{-1}\vec{e}_i = \vec{c}_i$$

unknown

$$\Rightarrow A\vec{c}_i = \vec{e}_i$$ (solve this equation for $\vec{c}_i$)

The construction of the inverse is equivalent to solving an equation!!

Linear Operators

Solve $\mathcal{L}[f]=b, \Rightarrow f=\mathcal{L}^{-1}[b]$.

To find $\mathcal{L}^{-1}$ need to compute

$\mathcal{L}^{-1}$[basis vector] = column vector

$\Rightarrow$ basis vector = $\mathcal{L}$[column vector]

* What is the basis vector??
* What do we mean by column vector??

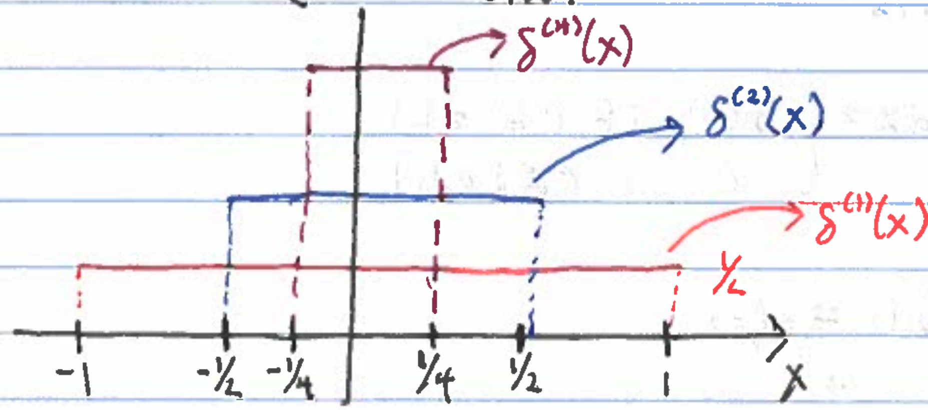
Operators on Function Spaces

	$\mathcal{L}=\frac{d^2}{dx^2}$	$A\in\mathbb{R}^{n\times n}$
Domain	C^2	$\mathbb{R}^n$
Range	C^0	$\mathbb{R}^2$
Basis Vector	function?	$\vec{e}_n$
Columns	function?	$\vec{c}_i$

	$f:\mathbb{R}\to\mathbb{R}$	$\vec{v}:\mathbb{N}\to\mathbb{R}$
Domain	$\mathbb{R}$	$\mathbb{N}$
Range	$\mathbb{R}$	$\mathbb{R}$
Basis Vector	$\delta_x(f)=f(x)$ $\Rightarrow\langle\delta_x,f\rangle=f(x)$	$\vec{e}_i(\vec{v})=v_i$ $\Rightarrow\langle\vec{e}_i,\vec{v}\rangle=v_i$

Delta Sequence

$$\delta_n(x) = \begin{cases} n/2, & -1/n < x < 1/n \\ 0 & \text{o.w.} \end{cases}$$



1. $$\lim_{n\to\infty} \delta_n(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

2. $$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1$$
$$\Rightarrow \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)\,dx = 1$$

3. $$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx = 0$$

Proof of Property 3:

$$f(0) = \int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx$$

$$\Rightarrow \left| f(0) - \int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx \right| = \left| \int_{-1/n}^{1/n} n/2\,(f(0) - f(x))\,dx \right|$$
$$\leq \int_{-1/n}^{1/n} n/2\, |f(0) - f(x)|\,dx$$

$$= \int_{-1/n}^{1/n} \frac{n}{2} |f'(c(x))| \cdot |x|\,dx$$
$$\leq \int_{-1/n}^{1/n} n/2 \cdot M \cdot 1/n\,dx$$
$$= M/n$$

where $M \equiv \max|f'(x)|$. By squeeze theorem.

$$\lim \int_{-\infty}^{\infty} \delta_n(x) f(x)\,dx = f(0)$$

Calculus of Delta Function

1. $\delta(x)$ is understood as a limit
$$\delta(x) \text{"="} \lim_{n\to\infty} \delta_n(x)$$

2. $$\int_a^b \delta(x) f(x)\,dx = \begin{cases} f(0), & \text{if } 0 \in [a,b] \\ 0, & \text{if } 0 \notin [a,b] \end{cases}$$

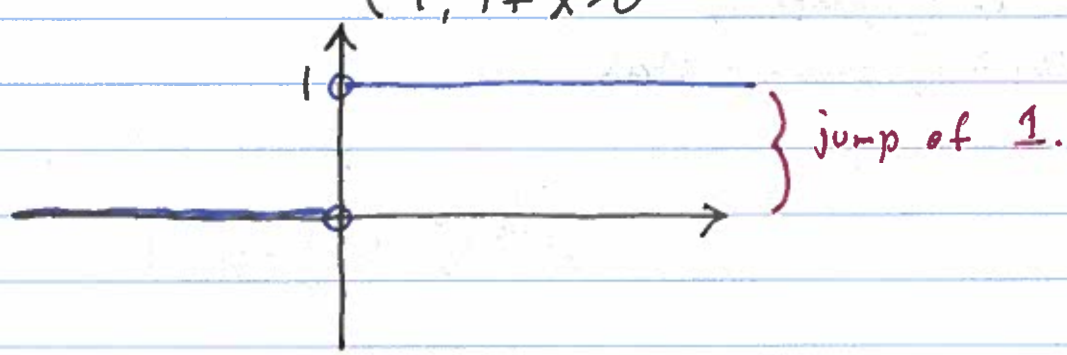
3. $$\int_{-\infty}^{\infty} \delta(x-y) f(x)\,dx = f(y)$$

Examples:

1. Let $h(x) = \delta(x) - 3\,\delta(x-1)$
$$\Rightarrow \langle h, f\rangle = \int_{-\infty}^{\infty} \big(\delta(x) - 3\delta(x-1)\big) f(x)\,dx$$
$$= f(0) - 3f(1).$$

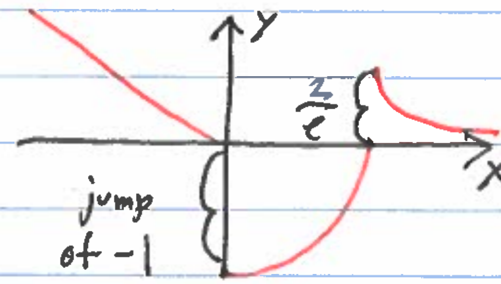
2. $x^3\delta(x) = 0$ since $\langle x^3\delta(x), f(x)\rangle = \langle \delta(x), x^3 f(x)\rangle = 0^3 \cdot f(0) = 0.$

3. The antiderivative:
$$\int_{-\infty}^{x} \delta(y)\,dy = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases} = \sigma(x)$$



Example:

$$f(x)=\begin{cases}-x, & x<0\\ x^2-1, & 0<x<1\\ 2e^{-x}, & x>1\end{cases}$$



$$\Rightarrow f'(x)=-\delta(x)+\frac{2}{e}\delta(x-1)+\begin{cases}-1, & x<0\\ 2x, & 0<x<1\\ -2e^{-x}, & x>1\end{cases}$$

Example:

Simplify the following

1. $f(x)\delta(x-a)=f(a)\delta(x-a)$
2. $f(x)\delta(1-x^2)$

$$\langle f(x)\delta(1-x^2), g(x)\rangle=\int_{-\infty}^{\infty} f(x)g(x)\delta(1-x^2)\,dx$$

$$=\int_{-\infty}^{0} f(x)g(x)\delta(1-x^2)\,dx+\int_{0}^{\infty} f(x)g(x)\delta(1-x^2)\,dx$$

Let $u=1-x^2$, $v=1-x^2$

$\Downarrow$ $\Downarrow$

$x=-\sqrt{1-u}$ $\quad$ $x=\sqrt{1-v}$

$du=-2x\,dx$ $\quad$ $dv=-2x\,dx$

$$\Rightarrow \langle f(x)\delta(1-x^2), g(x)\rangle=\int_{-\infty}^{1}\frac{f(-\sqrt{1-u})\,g(-\sqrt{1-u})\,\delta(u)\,du}{2(\sqrt{1-u})}$$

$$+\int_{1}^{-\infty}\frac{f(\sqrt{1-v})\,g(\sqrt{1-v})\,\delta(v)\,dv}{-2\sqrt{1-v}}$$

$$=\frac{1}{2}f(-1)g(-1)+\frac{1}{2}f(1)g(1)$$

$$\Rightarrow f(x)\delta(1-x^2)=\frac{1}{2}f(-1)\delta(x+1)+\frac{1}{2}f(1)\delta(x-1).$$

MTH 352/652: Homework #9

Due Date: April 19, 2024

1 Problems for Everyone

1. pg. 273-274 #7.1.1-7.1.4, #7.1.15, 7.1.16.
2. Compute $\mathcal{F}\left[xe^{-ax^2}\right](k)$, where $a > 0$ is a constant.
3. Given that

$$\mathcal{F}\left[xe^{-|x|}\right](k) = -\frac{4ik}{(1+k^2)^2},$$

find

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right].$$

Homework #9

pg. 273, #7.1.1

Find the Fourier transforms of the following functions

(a) $e^{-(x+4)^2}$

(c) $\begin{cases} x, & |x|\le 1 \\ 0, & \text{o.w.} \end{cases}$

(d) $\begin{cases} e^{-2x}, & x\ge 0 \\ e^{3x}, & x\le 0 \end{cases}$

(f) $\begin{cases} e^{-x}\sin(x), & x\ge 0 \\ 0, & x\le 0. \end{cases}$

Solution:

(a) $$\mathcal{F}[e^{-(x+4)^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x+4)^2} e^{-ikx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2} e^{-ik(u-4)}\,dx$$
$$= e^{4ik}\,\mathcal{F}[e^{-u^2}]$$
$$= e^{4ik}\frac{1}{\sqrt{2}}e^{-k^2/4}.$$

(c) $$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} x e^{-ikx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} x(\cos(kx) - i\sin(kx))\,dx$$
$$= \frac{-i2}{\sqrt{2\pi}}\int_{0}^{1} x\sin(kx)\,dx$$
$$= -i\sqrt{\frac{2}{\pi}}\left(-\frac{x\cos(kx)}{k}\Big|_0^1 + \int_0^1 \frac{\cos(kx)}{k}\,dx\right)$$
$$= i\sqrt{\frac{2}{\pi}}\frac{\cos(k)}{k} - i\sqrt{\frac{2}{\pi}}\frac{\sin(k)}{k^2}$$

(d) $\mathcal{F}[f] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{0} e^{3x}e^{-ikx}\,dx + \dfrac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-2x}e^{-ikx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(3-ik)x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(2+ik)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{3-ik} + \frac{1}{\sqrt{2\pi}}\frac{1}{2+ik}$$

(f) $\mathcal{F}[f] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{0}^{\infty} e^{-x}\sin(x)\,e^{-ikx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{e^{-x}(e^{ix}-e^{-ix})e^{-ikx}}{2i}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(1-i+ik)x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(1+i+ik)x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-i+ik} + \frac{1}{1+i+ik}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{(1+i)}{1+(k-1)^2}.$$

$\blacksquare$

pg. 274, # 7.1.15

Given that the Fourier transform of $f(x)$ is $\hat{f}(k)$, find the Fourier transform of $g(x) = f(ax+b)$, where $a, b \in \mathbb{R}$.

Solution:

$$\mathcal{F}[g] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(ax+b)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{a}\int_{-\infty}^{\infty} e^{-ik\left(\frac{u-b}{a}\right)} f(u)\,du$$

$$= \frac{e^{ikb/a}}{a}\hat{f}\left(\frac{k}{a}\right).$$

$\blacksquare$

~~pg. 274~~, #7.1.16

Let a be a real constant. Given the Fourier transform $\hat{f}(k)$ of $f(x)$, find the Fourier transform of (a) $f(x)e^{iax}$, (b) $f(x)\cos(ax)$, (c) $f(x)\sin(ax)$.

Solution:

(a) $$\mathcal{F}[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}e^{-ikx}f(x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(k-a)x}f(x)\,dx$$
$$= \hat{f}(k-a).$$

(b) $$\mathcal{F}[\cos(ax)f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{iax}+e^{-iax}}{2}\, e^{-ikx}f(x)\,dx$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{-i(k-a)x} + e^{-i(k+a)}\right)f(x)\,dx$$
$$= \frac{1}{2}\left(\hat{f}(k-a)+\hat{f}(k+a)\right)$$

(c) $$\mathcal{F}[\sin(ax)f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{iax}-e^{-iax}}{2i}\, e^{-ikx}f(x)\,dx$$
$$= \frac{1}{2i}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{-i(k-a)x} - e^{-i(k+a)x}\right)f(x)\,dx$$
$$= \frac{1}{2i}\left(\hat{f}(k-a)-\hat{f}(k+a)\right)$$

#2.

Compute $\mathcal{F}[xe^{-ax^2}]$, where $a>0$ is a constant.

Solution:

$$\mathcal{F}[xe^{-ax^2}] = \mathcal{F}\left[-\frac{1}{2a}\frac{d}{dx}(e^{-ax^2})\right]$$
$$= -\frac{1}{2a}(ik)\,\mathcal{F}[e^{-ax^2}]$$
$$= \frac{-ik}{2a}\,\frac{1}{\sqrt{2a}}\,e^{-k^2/4a}.$$
$$= \frac{-ik}{(2a)^{3/2}}\,e^{-k^2/4a}$$

■

#3.

Given that

$$\mathcal{F}[xe^{-|x|}] = -\frac{4ik}{(1+k^2)^2}$$

find

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right].$$

Solution:

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{(1+x^2)^2}e^{-ikx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-u}{(1+u^2)^2}e^{iku}\,du$$
$$= \frac{1}{4i}\,k e^{-|k|}$$
$$= -\frac{1}{4}\,ik e^{-|k|}$$

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