Lecture 16: Green's Functions

Recall to solve the equation

$$A\vec{x}=\vec{b}$$
$$\vec{x}=A^{-1}\vec{b}$$
$$=b_1A^{-1}\vec{e}_1+\cdots+b_nA^{-1}\vec{e}_n$$
$$=\sum_{i=1} b_i \underbrace{A^{-1}\vec{e}_i}_{i\text{-th column of } A^{-1}.}$$

We need to solve the equation

$$A\vec{c}_i=\vec{e}_i.$$

Example:

Solve the equation

$$\begin{cases} u''(x)=f(x) \\ u(0)=u(1)=0 \end{cases}$$

Let $\mathcal{L}=\frac{d^2}{dx^2}$, with B.C. $u(0)=u(1)=0$.

$\Rightarrow u(x)=\mathcal{L}^{-1}[f]$.

We need to find $\mathcal{L}^{-1}$.

Idea: Solve

$$u''(x)=\underbrace{\delta(x-y)}_{\text{basis function centered at } y.}$$

1. $u''(x)=0$, $x\neq 0$. (Properties of δ-function)
2. $\lim_{x\to y^+} u'(x)-\lim_{x\to y^-} u'(x)=1$ (Jump condition)
3. $\lim_{x\to y^+} u(x)=\lim_{x\to y^-} u(x)$ (Continuity)
4. $u(0)=u(1)=0$ (Boundary Conditions)

The generic solution is of the form

$$u(x) = \begin{cases} Ax+B, & x<y \\ Cx+D, & x>y \end{cases}$$

(Satisfies Property 1)

Property 4 implies:

$B=0$

$C+D=0 \Rightarrow D=-C$

Property 2 implies:

$C-A=1$

$\Rightarrow C=1+A$

Property 3 implies:

$Ay+B=Cy+D$

$\Rightarrow Ay=(1+A)y-1-A$

$\Rightarrow A=y-1$

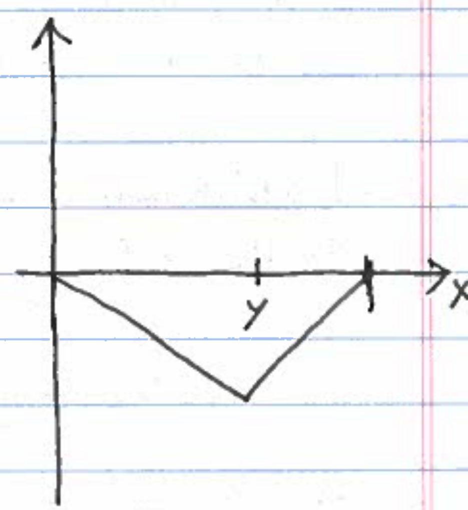
$C=y$

$D=-y$

The Green's function is this solution:

$$G(x,y) = \begin{cases} (y-1)x, & x<y \\ yx-y & x>y \end{cases}$$

$$= \begin{cases} (y-1)x & x<y \\ y(x-1) & x>y \end{cases}$$



This tells us that the solution is given by

$$u(x) = \int_0^1 G(x,y) f(y) dy$$

$$\Rightarrow \boxed{u(x) = \int_0^x y(x-1) f(y) dy + \int_x^1 (y-1) x f(y) dy}$$

Check:

Recall:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y) dy = \frac{d}{dx} F(x, b(x)) - F(x, a(x))$$

$$= \frac{\partial F}{\partial x}\Big|_{y=b(x)} + \frac{\partial F}{\partial y}\Big|_{y=b(x)} \frac{db}{dx} - \frac{\partial F}{\partial x}\Big|_{y=a(x)} - \frac{\partial F}{\partial y}\Big|_{y=a(x)} \frac{da}{dx}$$

$$= \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x} dy + f(x, b(x))\frac{db}{dx} - f(x, a(x))\frac{da}{dx}$$

Therefore,

$$u'(x) = \int_0^x y f(y) dy + \cancel{x(x-1)f(x)} + \int_x^1 (y-1) f(y) dy - \cancel{(x-1)x f(x)}$$

$$= \int_0^1 y f(y) dy - \int_x^1 f(y) dy$$

$$\Rightarrow u''(x) = \cancelto{0}{\frac{d}{dx}\int_0^1 y f(y) dy} - \frac{d}{dx}\int_x^1 f(y) dy$$

$$= f(x)$$

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