Lecture 2: Linear Operators

Linearity

A linear operator L is an operation on functions u, v satisfying

$$L[u+v] = L[u] + L[v],$$
$$L[cu] = cL[u],$$

where c is a constant.

Examples:

1. $L[u] = \dfrac{\partial u}{\partial x}$, other notation: $L = \partial_x$
2. $L[u] = \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial t}$, other notation: $L = \partial_x - \partial_t$
3. $L[u] = \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2}$, other notation: $L = \partial_t - \partial_x^2 - \partial_y^2$

Definition - A homogeneous linear differential equation has the form

$$L[u] = 0$$

where L is a differential operator.

Example:

Consider the PDE

$$u_{tt} - k(x)u_t = u_{xx}.$$

This is a homogeneous linear differential equation with

$$L = \partial_{tt} - k(x)\partial_t - \partial_{xx}$$

Theorem (Principle of Linear Superposition) - If u_1, u_2 solve $L[u]=0$ then u_1+u_2 and cu_1 are also solutions.

proof:

Suppose u_1, u_2 solve $L[u]=0$. Therefore,

1. $L[u_1+u_2]=L[u_1]+L[u_2]=0+0=0.$
2. $L[cu_1]=cL[u_1]=c\cdot 0=0.$

Example

We know that

$$u_1(t,x)=t+\tfrac{1}{2}x^2 \text{ and } u_2(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$$

solve $u_t=u_{xx}$. Therefore,

$$u(t,x)=c_1(t+\tfrac{1}{2}x^2)+c_2\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}.$$

also solves $u_t=u_{xx}$.

Theorem - The set of all solutions to a linear differential equation forms a vector space.

Definition - An inhomogeneous linear differential equation has the form

$$L[u]=f$$

Theorem - Let u^* be a particular solution to the inhomogeneous linear equation $L[u]=f$. The general solution is then u^*+u^H where u^H is a solution to the homogenous equation.

proof:

1. $L[u^*+u^H]=L[u^*]+L[u^H]=f+0=f.$
2. Suppose $\bar{u}$ is another solution, i.e., $L[\bar{u}]=f$. Then

$$L[\bar{u}-u]=L[\bar{u}]-L[u]=f-f=0$$

$\Rightarrow \bar{u}-u$ solves the homogeneous problem.