Lecture #4: Method of Characteristics

Nonuniform Transport

$$u_t + c(x)u_x = 0, \quad u(0,x) = f(x)$$

$\uparrow$ wave speed depends on position

Idea: Track the value of $u(t,x)$ along a curve $(t, x(t))$ in the t-x plane. Define

$$h(t) = u(t, x(t))$$

$$\Rightarrow \frac{dh}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

If $\frac{dx}{dt} = c(x)$ then

$$\frac{dh}{dt} = u_t + u_x c(x) = 0$$

$\Rightarrow u$ is constant along the curve.

Definition - The graph of a solution $x(t)$ to

$$\frac{dx}{dt} = c(x)$$

is a characteristic curve for $u_t + c(x)u_x = 0$

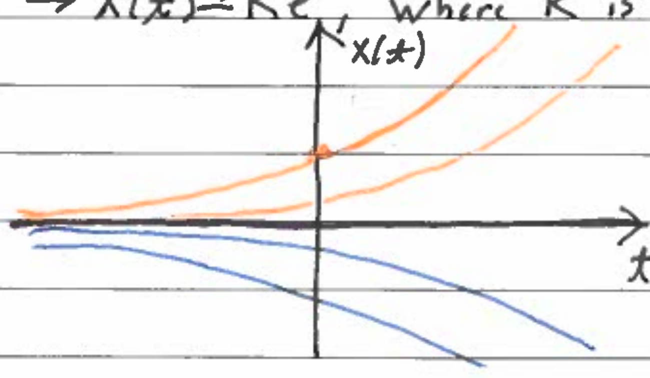
Example:

$$u_t + x u_x = 0, \quad u(0,x) = f(x)$$

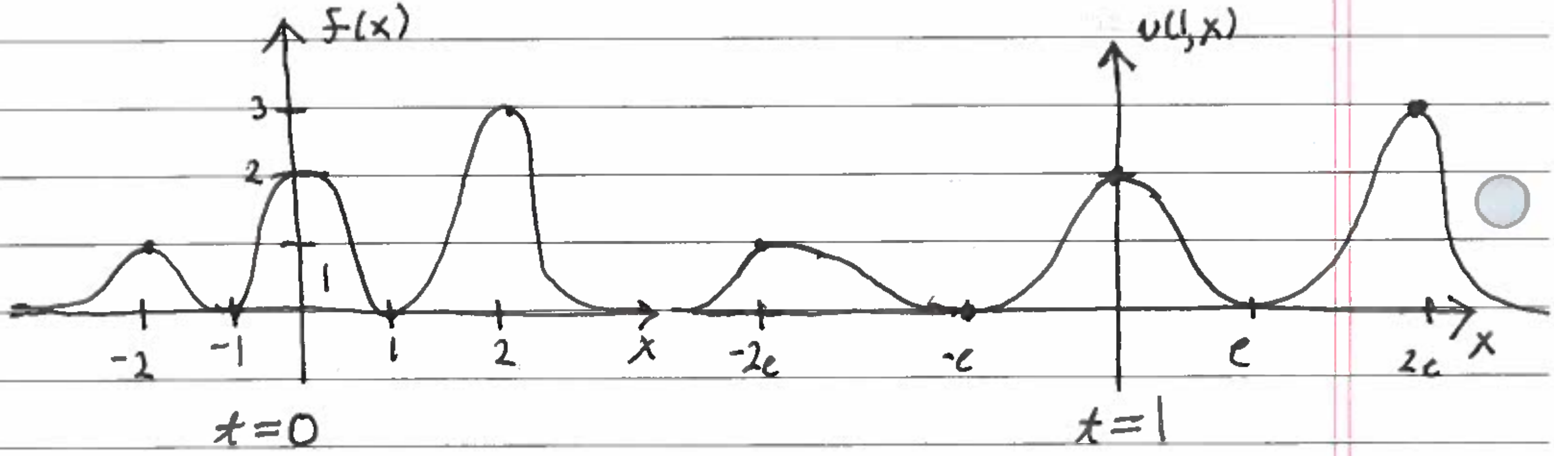
The characteristic curves are given by

$$\frac{dx}{dt} = x$$

$\Rightarrow x(t) = Ke^t$, where K is a constant.



$\rightarrow$ characteristic curves along which $u(t,x)$ is constant.



Suppose we want to find a general solution. We know that is constant along curves of the form

$$x(t) = ce^{t}$$

$$\Rightarrow \ln(|x|) - t = C \quad \text{(equation for contours)}$$

The generic solution is a function of $z = \ln(|x|) - t$:

$$u(t,x) = g(\ln(|x|) - t).$$

Applying boundary conditions

$$u(0,x) = g(\ln(|x|)) = f(x)$$

$$\Rightarrow g(x) = f(e^{x})$$

Therefore,

$$u(t,x) = f(e^{\ln(|x|) - t})$$

$$= f(xe^{-t}).$$

Check:

$$u_t = -xe^{-t} f'(xe^{-t})$$

$$u_x = e^{-t} f(xe^{-t})$$

Therefore, $u_t + xu_x = 0$. Also,

$$u(0,x) = f(xe^{0}) = f(x).$$

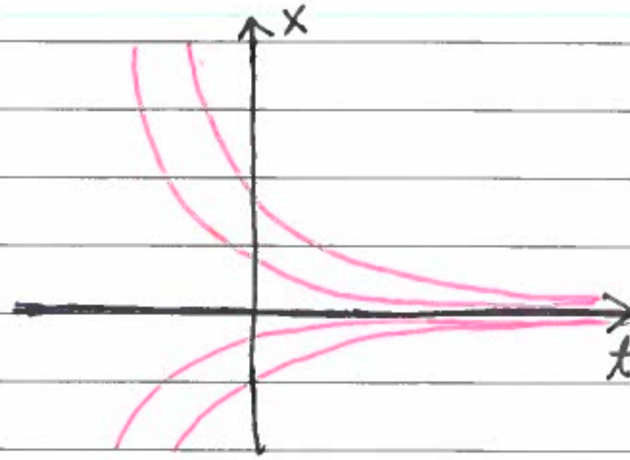
Example:

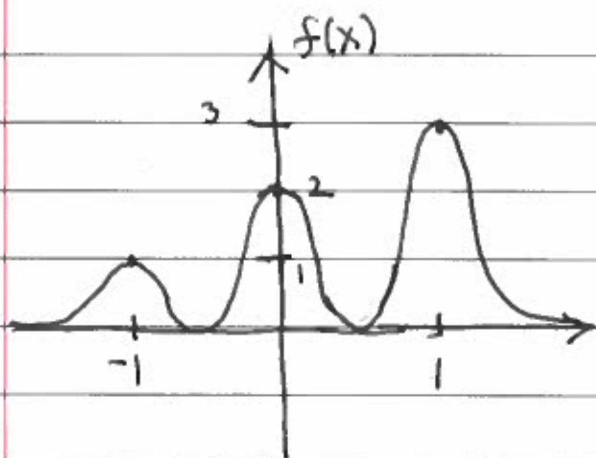
$u_t - x^3 u_x = 0, \quad u(0,x) = f(x)$

$$\frac{dx}{dt} = -x^3$$

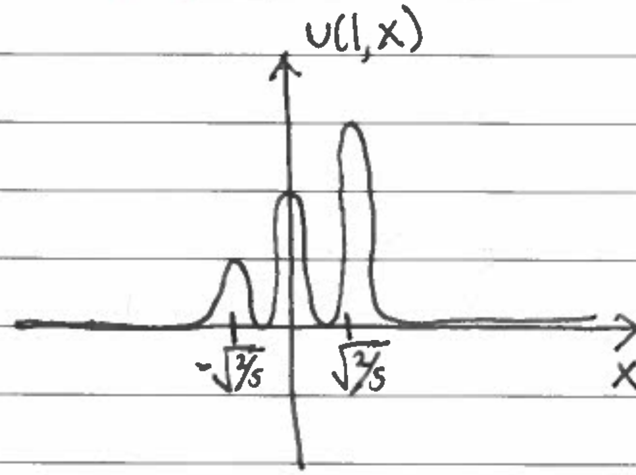
$$\Rightarrow \frac{1}{2x^2} = t + c$$

$$\Rightarrow x(t) = \pm \frac{1}{\sqrt{2t+c}}.$$





$t=0$



$t=1$

- At $t=0$, $x=-1 \Rightarrow u=1$. This lies along the charecteristic

  $$\frac{1}{2} = c$$

  At $t=1$, $x=-1$ is mapped to

  $$x(1) = \frac{-1}{\sqrt{2+1/2}} = -\sqrt{\frac{2}{5}}$$

- At $t=0$, $x=1 \Rightarrow u=3$. This lies along the characteristic

  $$\frac{1}{2} = c$$

  At $t=1$, $x=1$ is mapped to

  $$x(1) = \frac{1}{\sqrt{2+1/2}} = \sqrt{\frac{2}{5}}$$

Example:

$$u_t + t u_x = -u$$

Characteristic coordinate satisfies

$$\frac{dx}{dt} = t$$

$$\Rightarrow x(t) = \frac{t^2}{2} + c$$

Let $z = x - t^2/2$, $\tau = t$.

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = -t\frac{\partial u}{\partial z} + \frac{\partial u}{\partial \tau}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial u}{\partial z}$$

$$\Rightarrow u_\tau = -u$$

$$\Rightarrow u(\tau, z) = g(z)e^{-\tau}$$

$$\Rightarrow u(t, x) = g(x - t^2/2)e^{-t}.$$