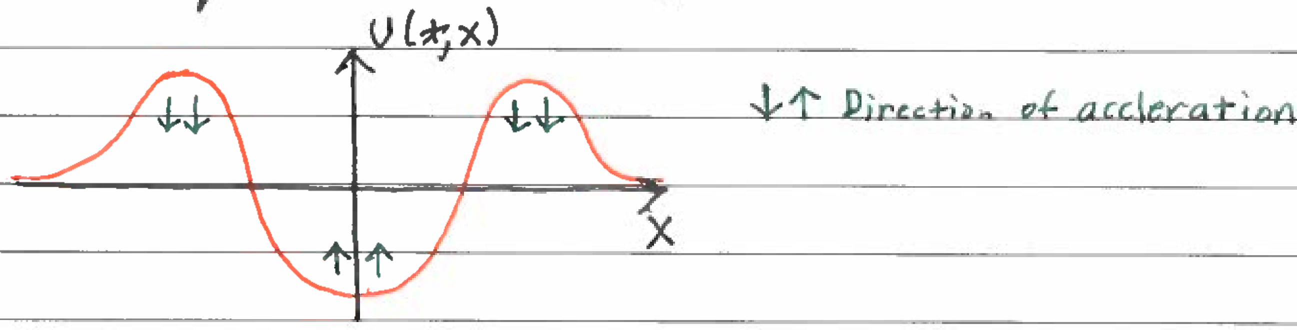
Lecture 5: The Wave Equation - d'Alembert's Formula

$$\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{accleration}} = c^2 \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{curvature}} \quad , \; u(t,x) \text{ - displacement of medium}$$

$u(0,x) = f(x)$ (Initial Displacement)

$u_t(0,x) = g(x)$ (Initial Velocity)



d'Alemberts Solution:

The wave operator $\Box$ is defined by

$$\Box u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}$$

which can be factored as:

$$\Box u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

$$= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u$$

$\Rightarrow$ If $\dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = 0$ or $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$

then $\Box u = 0$.

Consequently, by Lecture 3 and linearity it follows that

$$u(t,x) = p(x-ct) + q(x+ct)$$

is a solution.

$$\underbrace{p(x-ct)}_{\text{right travelling wave}} \quad , \quad \underbrace{q(x+ct)}_{\text{left travelling wave}}$$

Initial Value Problem:

Return to the equation

$$u_{tt} = c^2 u_{xx}$$
$$u(0,x) = f(x)$$
$$u_t(0,x) = g(x)$$

We know $u(t,x) = p(x-ct) + q(x+ct)$.

$$\Rightarrow p(x) + q(x) = f(x)$$
$$-cp'(x) + cq'(x) = g(x)$$

$$\Rightarrow cp'(x) + cq'(x) = cf'(x)$$
$$-cp'(x) + cq'(x) = g(x)$$

$$\Rightarrow q'(x) = \frac{1}{2} f'(x) + \frac{g(x)}{2c}$$

$$\Rightarrow q(x) = \frac{1}{2} f(x) + \int_0^x \frac{g(z)}{2c}\, dz, \quad p(z) = \frac{1}{2} f(x) - \int_0^x \frac{g(z)}{2c}\, dz$$

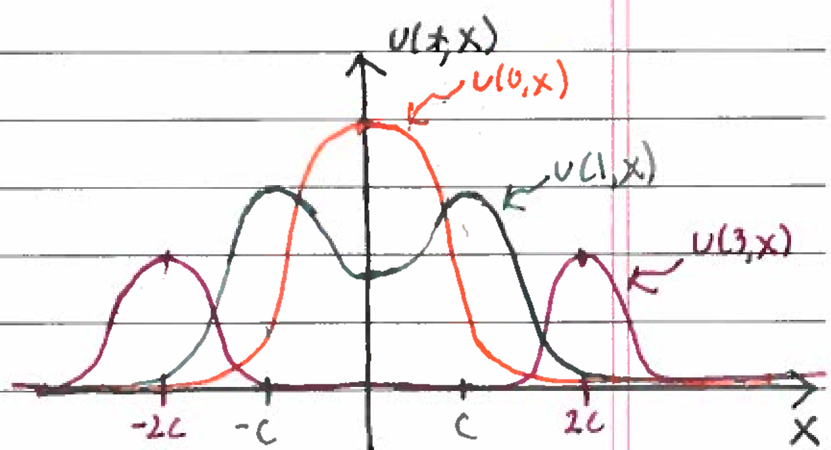
Therefore,

$$\boxed{u(t,x) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\, dz}$$

Example:

$$u_{tt} = c^2 u_{xx}$$
$$u(0,x) = e^{-x^2}$$
$$u_t(0,x) = 0$$

$$\Rightarrow u(t,x) = \frac{1}{2} e^{-(x-ct)^2} + \frac{1}{2} e^{-(x+ct)^2}$$



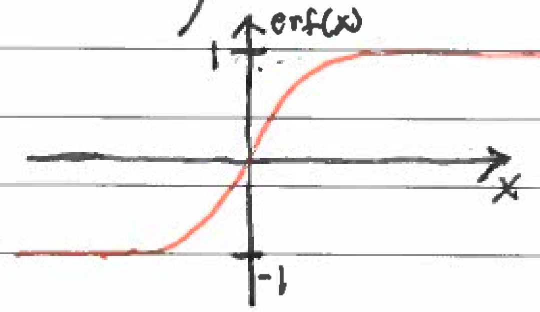
(Solution Plotted in Mathematica)

Example:

$$u_{tt} = c^2 u_{xx}$$
$$u(0,x) = 0$$
$$u_t(0,x) = e^{-x^2}$$

$$\Rightarrow u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} e^{-s^2}\,ds,$$
$$= \frac{\sqrt{\pi}}{4c}\Big(\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)\Big),$$

Where
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\,ds.$$



Uniqueness of Solutions

Let $u(t,x)$ solve

$$u_{tt} = c^2 u_{xx}$$
$$u(0,x) = f(x) \qquad *$$
$$u_t(0,x) = g(x)$$

Define the energy by

$$E(t) = \int_{-\infty}^{\infty}\left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)dx$$

$$\Rightarrow \frac{dE}{dt} = \int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right)dx$$

$$= \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\right)dx$$

u vanishes at infinity by assumption.

$$= \int_{-\infty}^{\infty}\frac{\partial u}{\partial t}\left(c^2\frac{\partial^2 u}{\partial x^2} - c^2\frac{\partial^2 u}{\partial x^2}\right)dx$$

$$= 0.$$

Therefore $E(t)$ is a constant given by

$$E(t) = E(0) = \int_{-\infty}^{\infty} \left(\frac{1}{2} g(x)^2 + \frac{c^2}{2} f'(x)^2 \right) dx.$$

Now, suppose u_1, u_2 solve $*$. Then, if we let $v = u_1 - u_2$ then

$$v_{tt} = c^2 v_{xx}$$
$$v(0,x) = 0$$
$$v_t(0,x) = 0$$

With associated energy $E=0$. Consequently, $v=0$ which implies

$$u_1 = u_2.$$