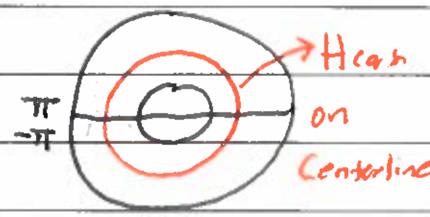
# Lecture 6: Heated Ring

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial \theta^2} \\ u(0,\theta) = f(\theta) & \} \text{ Initial Conditions} \\ u(t,\pi) = u(t,-\pi) \\ u_x(t,\pi) = u_x(t,-\pi) \end{cases} \Bigg\} \text{ Boundary Conditions}$$




## How do we solve?

Guess: $u(t,\theta) = T(t)\Theta(\theta)$

$\Rightarrow T'\Theta = T\Theta''$

$\Rightarrow \dfrac{T'}{T} = \dfrac{\Theta''}{\Theta} = \lambda$ → must be a constant

$\Rightarrow * \Theta'' = \lambda\Theta, \quad T' = \lambda T *$

$\Rightarrow \Theta(\theta) = \underbrace{Ae^{\sqrt{\lambda}\theta} + Be^{-\sqrt{\lambda}\theta}}_{\text{eigenfunction of } \frac{d^2}{d\theta^2}}.$

## Case 1 ($\lambda > 0$):

Boundary conditions imply

$\Theta(\pi) = \Theta(-\pi) \Rightarrow Ae^{\sqrt{\lambda}\pi} + Be^{-\sqrt{\lambda}\pi} = Ae^{-\sqrt{\lambda}\pi} + Be^{\sqrt{\lambda}\pi}$

$\Theta'(\pi) = \Theta'(-\pi) \Rightarrow Ae^{\sqrt{\lambda}\pi} - Be^{-\sqrt{\lambda}\pi} = Ae^{-\sqrt{\lambda}\pi} - Be^{\sqrt{\lambda}\pi}$

$\Rightarrow$ This is only possible if $A = B = 0$.

## Case 2 ($\lambda = 0$):

$\Theta'' = 0$

$\Rightarrow \Theta = A\theta + b$

Boundary conditions imply $A = 0$

$\Rightarrow \Theta = b$

$\Theta = b$ satisfies the PDE and boundary conditions but not the initial conditions.

Case 3 ($\lambda < 0$):

Let $\lambda = -\omega^2$. It follows that:

$$\Theta'' = -\omega^2 \Theta$$

$$\Rightarrow \Theta(\theta) = A\cos(\omega\theta) + B\sin(\omega\theta)$$

Boundary conditions

$$A\cos(\omega\pi) + B\sin(\omega\pi) = A\cos(\omega\pi) - B\sin(\omega\pi)$$

$$-A\sin(\omega\pi) + B\cos(\omega\pi) = A\sin(\omega\pi) + B\cos(\omega\pi)$$

$$\Rightarrow B\sin(\omega\pi) = 0$$

$$A\cos(\omega\pi) = 0$$

If $\omega \neq n \in \mathbb{N}$, $A = B = 0$.

If $\omega = n \in \mathbb{N}$, $A, B$ are free variables.

Therefore, for $n \in \mathbb{N}$,

$$u(t,\theta) = e^{-nt}(A\cos(n\theta) + B\sin(n\theta))$$

Solves the PDE and boundary conditions but not the initial conditions.

General Solution:

Take an infinite linear combination:

$$\boxed{u(t,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos(n\theta) + \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(n\theta)}$$

Initial conditions imply:

$$u(0,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta).$$

$a_0$:

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} a_0 + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi} \cos(n\theta)\,d\theta}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\sin(n\theta)}_{0}$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta}$$

$a_n$:

$$\int_{-\pi}^{\pi} \cos(m\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} a_0 \underbrace{\cos(m\theta)\,d\theta}_{0} + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(m\theta)\cos(n\theta)\,d\theta + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \underbrace{\sin(n\theta)\cos(m\theta)\,d\theta}_{0}$$

Now,

$$\cos(m\theta)\cos(n\theta) = \frac{1}{2}\cos((m-n)\theta) + \frac{1}{2}\cos((m+n)\theta)$$

$$\Rightarrow \int_{-\pi}^{\pi} \cos(m\theta)\cos(n\theta)\,d\theta = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

Therefore,

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta}$$

$b_n$:

$$\int_{-\pi}^{\pi} \sin(m\theta) f(\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} a_0 \underbrace{\sin(m\theta)\,d\theta}_{0} + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \underbrace{\cos(n\theta)\sin(m\theta)\,d\theta}_{0} + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(n\theta)\sin(m\theta)\,d\theta$$

$$= \pi b_m$$

Therefore,

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta}$$

## Example

Solve $u_t = u_{xx}$

$u(0,x) = \cos^2(x)$

$u(t,\pi) = u(t,-\pi)$

$u_x(t,\pi) = u_x(t,-\pi)$

A solution is

$$u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos(nx) + \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx)$$

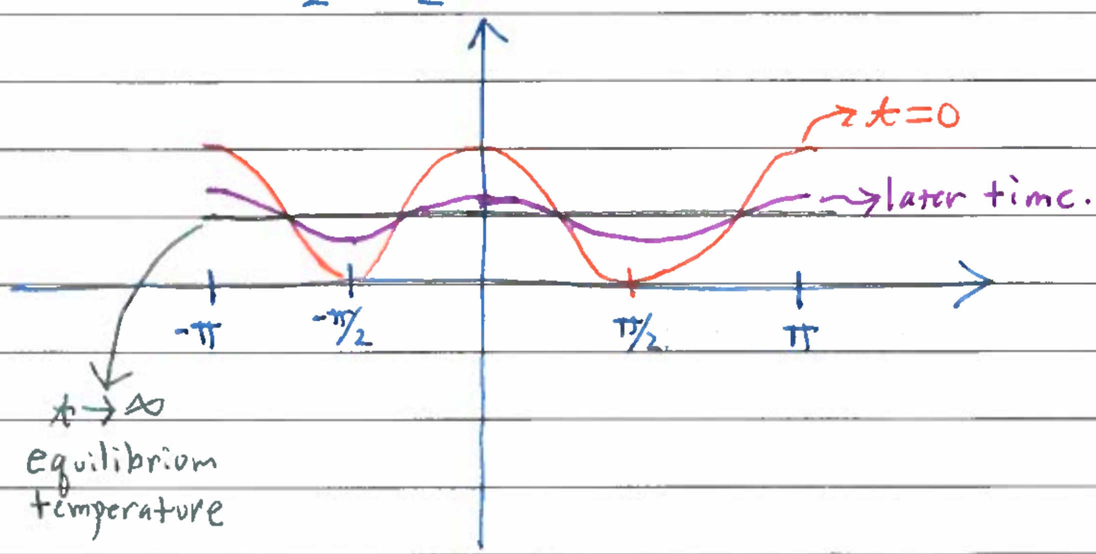
Boundary conditions imply

$$\cos^2(x) = u(0,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow \frac{1+\cos(2x)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$\Rightarrow a_0 = 1$, $a_2 = 1/2$, all other constants are $0$.

$$\Rightarrow u(t,x) = \frac{1}{2} + \frac{1}{2} e^{-4t}\cos(2x).$$



Properties:

1. $H(t)=\int_{-\pi}^{\pi} u(t,x)dx$

$$\Rightarrow \frac{dH}{dt}=\int_{-\pi}^{\pi} u_t(t,x)dx$$
$$=\int_{-\pi}^{\pi} u_{xx}(t,x)dx$$
$$=u_x\Big|_{-\pi}^{\pi}$$
$$=u_x(t,\pi)-u_x(t,-\pi)$$
$$=0.$$

The total heat is conserved and is given by

$$H(t)=H(0)$$
$$=\int_{-\pi}^{\pi} u(0,x)dx$$
$$=\int_{-\pi}^{\pi} f(x)\,dx$$
$$=\pi a_0$$

2. Let $E(t)=\frac{1}{2}\int_{-\pi}^{\pi} u^2(t,x)dx$

$$\Rightarrow \frac{dE}{dt}=\int_{-\pi}^{\pi} u_t u\,dx$$
$$=\int_{-\pi}^{\pi} u_{xx} u\,dx$$
$$=u_x u\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi} u_x^2 dx$$
$$=\cancel{u_x(t,\pi)u(t,\pi)-u_x(t,-\pi)u(t,-\pi)}-\int_{-\pi}^{\pi} u_x^2 dx$$
$$=-\int_{-\pi}^{\pi} u_x^2 dx$$

$$\Rightarrow \frac{dE}{dt}\leq 0$$

The energy is decreasing in time.

3. Solutions are unique:

Suppose $u_1, u_2$ are solutions and let $v=u_1-u_2$. Therefore,

$$v_{tt}=v_{xx}$$
$$v(t,\pi)=v(t,-\pi)$$
$$v_x(t,\pi)=v_x(t,-\pi)$$
$$v(0,x)=0$$

We know that

1. $\frac{dE}{dt} \leq 0$
2. $E(t) \geq 0$
3. $E(0) = \int_{-\pi}^{\pi} v(0,x)^2 dx = \int_{-\pi}^{\pi} 0 \, dx = 0$

Consequently, for all $t$, $E(t) = 0$. Therefore, $v(t,x) = 0$ and thus $u_1 = u_2$.