Lecture 7: More on Fourier Series

Example:

Find the Fourier series for

$$f(x) = \sin^5(x).$$

on $[-\pi, \pi]$. We can use Euler's formula to simplify

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$$

$$\sin^5\theta = \frac{1}{(2i)^5}\left(e^{i\theta}-e^{-i\theta}\right)^5$$
$$= \frac{1}{32i}\left(e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}\right)$$
$$= \frac{1}{16}\sin(5\theta) - \frac{5}{16}\sin(3\theta) + \frac{5}{8}\sin(\theta)$$

$$\Rightarrow b_1 = \frac{5}{8}, \quad b_2 = -\frac{5}{16}, \quad b_3 = \frac{1}{16}$$

Example:

Find the Fourier series for

$$f(x) = \sin(\sqrt{2}x)$$

on the interval $[0, 2]$.

The orthogonal system for $[0, 2\pi]$ is

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$$

We compress these functions by the factor $1/\pi$ to get the following orthogonal system:

$\{1, \cos(\pi x), \sin(\pi x), \cos(2\pi x), \sin(2\pi x), \ldots\}$

$$\Rightarrow \sin(\sqrt{2}x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Coefficients:

1. $\int_0^2 \sin(\sqrt{2}x)\,dx = \int_0^2 \frac{a_0}{2}\,dx = a_0$

$$\Rightarrow \left.\frac{-\cos(\sqrt{2}x)}{\sqrt{2}}\right|_0^2 = a_0$$

$$\Rightarrow a_0 = \frac{1}{\sqrt{2}}(1 - \cos(2\sqrt{2}))$$

2. $\int_0^2 \sin(\sqrt{2}x)\sin(n\pi x)\,dx = \int_0^2 b_n \sin^2(n\pi x)\,dx$

$$\Rightarrow \int_0^2 \left(\frac{e^{i\sqrt{2}x} - e^{-i\sqrt{2}x}}{2i}\right)\left(\frac{e^{in\pi x} - e^{-in\pi x}}{2i}\right)dx = b_n$$

$$\Rightarrow -\frac{1}{4}\int_0^2 \left(e^{i(\sqrt{2}+n\pi)x} - e^{i(\sqrt{2}-n\pi)x} - e^{-i(\sqrt{2}-in\pi)x} + e^{i(\sqrt{2}+n\pi)x}\right)dx = b_n$$

$$-\frac{1}{2}\int_0^2 \left(\cos((\sqrt{2}+n\pi)x) - \cos((\sqrt{2}-n\pi)x)\right)dx = b_n$$

$$\Rightarrow b_n = \frac{1}{2}\left(\frac{\sin(2(\sqrt{2}-n\pi)) - 1}{\sqrt{2}-n\pi} - \frac{\cos(2(\sqrt{2}+n\pi)) + 1}{\sqrt{2}+n\pi}\right)$$

3. $a_n = 0.$

Differentiation and Integration

1. Given

$$x \sim 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx),$$

What is the Fourier series of x^2?

$$\Rightarrow \frac{x^2}{2} \sim c + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+2}}{k^2} \cos(kx)$$

$$\Rightarrow x^2 \sim c + 4\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^2} \cos(kx)$$

Need to find c:

$$\int_{-\pi}^{\pi} x^2 dx = 2\pi c$$

$$\Rightarrow c = \pi^2/3$$

$$\Rightarrow x^2 \sim \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^2} \cos(kx).$$

2. What is the Fourier series of x^3?

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^2} \cos(kx)$$

$$\Rightarrow \frac{x^3}{3} \sim c + \frac{\pi^2}{3}x + 4\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^3} \sin(kx)$$

$$\Rightarrow x^3 \sim c + \pi^2 x + 12\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k^3} \sin(kx)$$

$$\sim c + \sum_{k=1}^{\infty} \left(\frac{2\pi^2(-1)^{k+1}}{k} + \frac{12(-1)^{k}}{k} \right) \sin(kx)$$

Since $\int_{-\pi}^{\pi} x^3 dx = 0$ it follows that $c = 0$.

3. What about a Fourier series for $f(x)=1$. If we just differentiate we have

$$1 \sim 2\sum_{k=1}^{\infty}(-1)^{k+1}\cos(kx),$$

which is not remotely true!!

*If $\tilde{f} \in C^1$ then the Fourier series for $f'(x)$ is the term by term derivative for the Fourier series of $f(x)$.