Lecture 8: Finite Difference Solutions to Heat Equation

Finite Difference Approximations

1. $$u'(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$\Rightarrow u'(x) \approx \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

How accurate is this? Taylor expand with respect to Δx:

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \tfrac{1}{2} u''(\xi)\Delta x^2$$

$$\Rightarrow u'(x) = \frac{u(x+\Delta x) - u(x)}{\Delta x} - \frac{1}{2} u''(\xi)\Delta x$$

Therefore

$$\boxed{\left| \frac{u(x+\Delta x) - u(x)}{\Delta x} - u'(x) \right| < C\Delta x \quad C = \max \tfrac{1}{2} |u''(\xi)|}$$

The approximation is first order in Δx:

$$\boxed{u'(x) = \frac{u(x+\Delta x) - u(x)}{\Delta x} + \mathcal{O}(\Delta x)}$$

2. To derive a second order approximation

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \tfrac{1}{2} u''(x)\Delta x^2 + \mathcal{O}(\Delta x^3)$$

$$u(x-\Delta x) = u(x) - u'(x)\Delta x + \tfrac{1}{2} u''(x)\Delta x^2 + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow u(x+\Delta x) - u(x-\Delta x) = 2u'(x)\Delta x + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow \boxed{u'(x) = \frac{u(x+\Delta x) - u(x-\Delta x)}{2} + \mathcal{O}(\Delta x^2)}$$

3. Derive a second order forward operator:

$$u(x+\Delta x)=u(x)+u'(x)\Delta x+\tfrac{1}{2}u''(x)\Delta x^2+\mathcal{O}(\Delta x^3)$$
$$u(x+2\Delta x)=u(x)+2u'(x)\Delta x+2u''(x)\Delta x^2+\mathcal{O}(\Delta x^3)$$

$$\Rightarrow 4u(x+\Delta x)-u(x+2\Delta x)=3u(x)+2u'(x)\Delta x+\mathcal{O}(\Delta x^3)$$
$$\Rightarrow \boxed{u'(x)=\frac{-3u(x)+4u(x+\Delta x)-u(x+2\Delta x)}{2\Delta x}+\mathcal{O}(\Delta x^2)}$$

4. Second order backward operator:

$$\boxed{u'(x)=\frac{u(x-2\Delta x)-4u(x-\Delta x)+3u(x)}{2\Delta x}+\mathcal{O}(\Delta x^2)}$$

5. Second order second derivative:

$$u(x+\Delta x)=u(x)+u'(x)\Delta x+\tfrac{1}{2}u''(x)\Delta x^2+\tfrac{1}{6}u'''(x)\Delta x^3+\mathcal{O}(\Delta x^4)$$
$$u(x-\Delta x)=u(x)-u'(x)\Delta x+\tfrac{1}{2}u''(x)\Delta x^2-\tfrac{1}{6}u'''(x)\Delta x^3+\mathcal{O}(\Delta x^4)$$

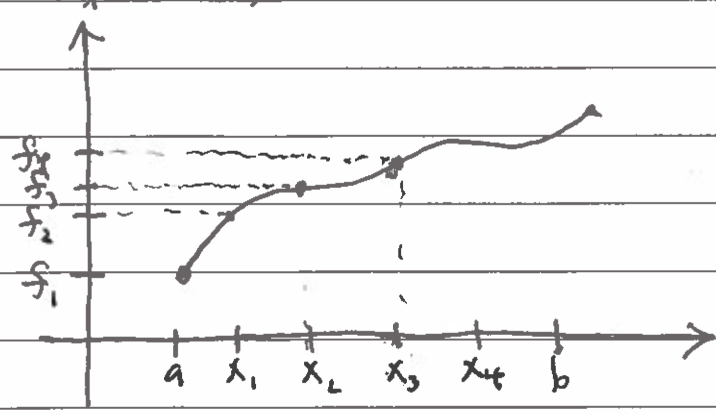
$$\Rightarrow u(x+\Delta x)+u(x-\Delta x)=2u(x)+u''(x)\Delta x^2+\mathcal{O}(\Delta x^4)$$
$$\Rightarrow \boxed{u''(x)=\frac{u(x-\Delta x)-2u(x)+u(x+\Delta x)}{\Delta x^2}+\mathcal{O}(\Delta x^2)}$$

Matrix Representation

Discretize $f(x)$:

$f_i=f(x_i)$,

$x_i=a+i\Delta x$



Define

$$f_1' = \frac{-3f_1 + 4f_2 - f_3}{2\Delta x}$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

$$f_n' = \frac{f_{n-2} - 4f_{n-1} + 3f_n}{2\Delta x}$$

Let

$$\vec{f}' = \begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Then

$$\boxed{\vec{f}' = D\vec{f}, \quad D = \frac{1}{2\Delta x}\begin{bmatrix} -3 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 3 \end{bmatrix}}$$

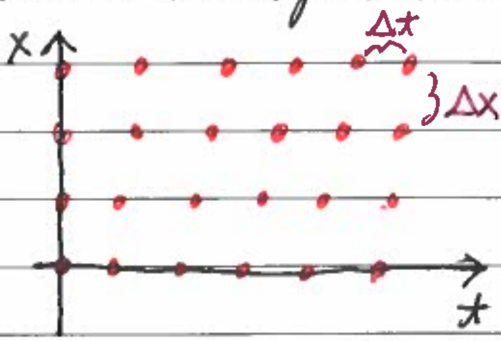
Finite Difference Solution to PDE

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

$$u(0,x) = f(x)$$

$$u(t,0) = u(t,L) = 0$$

Introduce a rectangular mesh of nodes $(t_i, x_j) \in \mathbb{R}^2$



$\Delta t = t_{j+1} - t_j$

$\Delta x = x_{j+1} - x_j$

$U_{ij} = u(t_j, x_i)$ → Columns of U are different representations of u at different times.

$$u_t(t_j, x_i) = \frac{u(t_{j+1}, x_i) - u(t_j, x_i)}{\Delta t} + O(\Delta t)$$

$$u_{xx}(t_j, x_i) = \frac{u(t_j, x_{i+1}) - 2u(t_j, x_i) + u(t_j, x_{i-1})}{\Delta x^2} + O(\Delta x^2)$$

$$\boxed{U_{i,j+1} = U_{i,j} + \frac{\gamma \Delta t}{\Delta x^2}\left(U_{i,j+1} - 2U_{i,j} + U_{i,j-1}\right)}$$

For stability we need

$$\frac{\gamma \Delta t}{\Delta x^2} < 1$$

$$\Rightarrow \boxed{\Delta t < \frac{1}{\gamma} \Delta x^2}$$