CONNECTED SUMS OF GORENSTEIN LOCAL RINGS

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To Gerson Levin, on his seventieth birthday.

ABSTRACT. Given surjective homomorphisms $R \to T \leftarrow S$ of local rings, and ideals in R and S that are isomorphic to some T-module V, the *connected sum* $R\#_T S$ is defined to be ring obtained by factoring out the diagonal image of V in the fiber product $R \times_T S$. When T is Cohen-Macaulay of dimension d and V is a canonical module of T, it is proved that if R and S are Gorenstein of dimension d, then so is $R\#_T S$. This result is used to study how closely an artinian ring can be approximated by a Gorenstein ring mapping onto it. When T is regular, it is shown that $R\#_T S$ almost never is a complete intersection ring. The proof uses a presentation of the cohomology algebra $\mathrm{Ext}^*_{R\#_k S}(k,k)$ as an amalgam of the algebras $\mathrm{Ext}^*_R(k,k)$ and $\mathrm{Ext}^*_S(k,k)$ over isomorphic polynomial subalgebras generated by one element of degree 2.

INTRODUCTION

We introduce, study, and apply a new construction of local Gorenstein rings.

The starting point is the classical fiber product $R \times_T S$ of a pair of surjective homomorphisms $\varepsilon_R \colon R \to T \leftarrow S : \varepsilon_S$ of local rings. It is well known that this ring is local, but until recently, little was known about its properties. In Proposition 1.7 we show that if R, S, and T are Cohen-Macaulay of dimension d, then so is $R \times_T S$, but this ring is Gorenstein only in trivial cases. When $\varepsilon_R = \varepsilon_S$, D'Anna [6] and Shapiro [22] proposed and partly proved a criterion for $R \times_T R$ to be Gorenstein. We complete and strengthen their results in Theorem 1.8: $R \times_T R$ Is Gorenstein if and only if R is Cohen-Macaulay and $\operatorname{Ker} \varepsilon_R$ is a canonical module for R.

Our main construction involves, in addition to the ring homomorphisms ε_R and ε_S, a T-module V and homomorphisms $\iota_R \colon V \to R$ of R-modules and $\iota_S \colon V \to S$ of S-modules, for the structures induced through ε_R and ε_S, respectively. When these maps satisfy $\varepsilon_R \iota_R = \varepsilon_S \iota_S$, we define a *connected sum* ring by the formula

$$R\#_T S = (R \times_T S)/\{(\iota_R(v), \iota_S(v)) \mid v \in V\}\,.$$

In case R, S, and T have dimension d (for some $d \geq 0$), R and S are Gorenstein, T is Cohen-Macaulay, and V is a canonical module for T, one can choose ι_R and ι_S to be isomorphisms onto $(0 : \operatorname{Ker}(\varepsilon_R))$ and $(0 : \operatorname{Ker}(\varepsilon_S))$, respectively. In Theorem 2.8 we prove that if $\varepsilon_R \iota_R = \varepsilon_S \iota_S$ holds, then $R\#_T S$ is Gorenstein of dimension d.

Much of the paper is concerned with Gorenstein rings of this form.

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As a first application, we study how efficiently an artinian local ring can be approximated by a Gorenstein artinian ring mapping onto it. One numerical measure of proximity is given by the Gorenstein colength of an artinian ring, introduced in [1]. We obtain new estimates for this invariant. We use them in the proof of Theorem 4.7 to remove a restrictive hypothesis from a result of Huneke and Vraciu [10], describing homomorphic images of Gorenstein local rings modulo their socles.

When $d = 0$ and T is a field, the construction of $R\#_T S$ mimics the expression for the cohomology algebra of a connected sum $M\#N$ of compact smooth manifolds M and N, in terms of the cohomology algebras of M and N; see Example 3.6. This analogy provides the name and the notation for connected sums of rings.

The topological analogy also suggests that connected sums may be used for classifying Cohen-Macaulay quotient rings of Gorenstein rings. The corresponding classification problem is, in a heuristic sense, dual to the one approached through Gorenstein linkage: Whereas linkage operates on the set of *Cohen-Macaulay quotients of a fixed Gorenstein ring R*, connected sums operate on the set of *Gorenstein rings with a fixed Cohen-Macaulay quotient ring T*.

This point of view raises the question of identifying those rings Q that are *indecomposable*, in the sense that an isomorphism $Q \cong R\#_T S$ implies $Q \cong R$ or $Q \cong S$. In Theorem 8.3 we show that if T is regular and Q is complete intersection, then either Q is indecomposable, or it is a connected sum of two quadratic hypersurface rings. The argument uses the structure of the algebra $\mathrm{Ext}^*_{R\#_T S}(T,T)$, when R and S are artinian and T is a field. In Theorem 7.3 we show that it is an amalgam of $\mathrm{Ext}^*_R(T,T)$ and $\mathrm{Ext}^*_S(T,T)$ over a polynomial T-subalgebra, generated by an element of degree 2. The machinery for the proof is fine-tuned in Sections 5 and 6.

1. Fiber products

The *fiber product* of a diagram of homomorphisms of commutative rings

$$
\begin{tikzcd}
R \arrow[dr, "\varepsilon_R"] & \\
& T \\
S \arrow[ur, "\varepsilon_S"'] &
\end{tikzcd}
\tag{1.0.1}
$$

is the subring of $R \times S$, defined by the formula

$$R \times_T S = \{(x,y) \in R \times S \mid \varepsilon_R(x) = \varepsilon_S(y)\}\,. \tag{1.0.2}$$

If $R \xleftarrow{\alpha_R} A \xrightarrow{\alpha_S} S$ are surjective homomorphisms of rings, then for $T = R \otimes_A S$, $\varepsilon_R(r) = r \otimes 1$, and $\varepsilon_S(s) = 1 \otimes s$ the map $a \mapsto (\alpha_R(a), \alpha_S(a))$ is a surjective homomorphism of rings $A \to R \times_T S$ with kernel $\mathrm{Ker}(\alpha_R) \cap \mathrm{Ker}(\alpha_S)$, whence

$$R \times_T S \cong A/(\mathrm{Ker}(\alpha_R) \cap \mathrm{Ker}(\alpha_S))\,. \tag{1.0.3}$$

In the sequel, the phrase $(Q, \mathfrak{q}, k)$ *is a local ring* means that Q is a commutative noetherian ring with unique maximal ideal $\mathfrak{q}$ and residue field $k = Q/\mathfrak{q}$.

The following setup and notation are in force for the rest of this section:

1.1. The rings in diagram (1.0.1) are local: $(R, \mathfrak{r}, k)$, $(S, \mathfrak{s}, k)$, and $(T, \mathfrak{t}, k)$.

The maps ε_R and ε_S are surjective; set $I = \mathrm{Ker}(\varepsilon_R)$ and $J = \mathrm{Ker}(\varepsilon_S)$, and also

$$P = R \times_T S\,.$$

Let η denote the inclusion of rings $P \to R \times S$, and let $R \leftarrow R \times S \to S$ be the canonical maps. Each (finite) module over R or S acquires a canonical structure of (finite) P-module through the composed homomorphisms of rings $R \leftarrow P \to S$ (finiteness is preserved because these maps are surjective).

The rings and ideals above are related through exact sequences of P-modules

$$0 \longrightarrow I \oplus J \longrightarrow R \oplus S \xrightarrow{\varepsilon_R \oplus \varepsilon_S} T \oplus T \longrightarrow 0 \tag{1.1.1}$$

$$0 \longrightarrow R \times_T S \xrightarrow{\eta} R \oplus S \xrightarrow{(\varepsilon_R, -\varepsilon_S)} T \longrightarrow 0 \tag{1.1.2}$$

A length count in the second sequence yields the relation

$$\ell(R \times_T S) + \ell(T) = \ell(R) + \ell(S)\,. \tag{1.1.3}$$

For completeness, we include a proof of the following result; see [8, 19.3.2.1].

Lemma 1.2. *The ring $R \times_T S$ is local, with maximal ideal $\mathfrak{p} = \mathfrak{r} \times_{\mathfrak{t}} \mathfrak{s}$.*

Proof. The rings R and S are quotients of P, so they are noetherian P-modules. Thus, the P-module $R \oplus S$ is noetherian, and hence so is its submodule P.

If (r, s) is in P, but not in $\mathfrak{r} \times_{\mathfrak{t}} \mathfrak{s}$, then r is not in $\mathfrak{r}$, so r is invertible in R. Since ε_S is surjective, there exists $s' \in S$ with $\varepsilon_S(s') = \varepsilon_R(r^{-1})$. One then has $\varepsilon_S(s's) = \varepsilon_R(r^{-1})\varepsilon_R(r) = 1$, so $a = s's$ is an invertible element of S. Now $(r^{-1}, a^{-1}s')$ is in P, and it satisfies $(r^{-1}, a^{-1}s')(r, s) = (r^{-1}r, a^{-1}s's) = (1, 1)$. $\square$

For any sequence $\boldsymbol{x}$ of elements of P and P-module M, we let $\mathrm{H}_n(\boldsymbol{x}, M)$ denote the nth homology module of the Koszul complex on $\boldsymbol{x}$ with coefficients in M.

Lemma 1.3. *When $\boldsymbol{x}$ is a T-regular sequence in $R \times_T S$ and $\overline{M}$ denotes $M/\boldsymbol{x}M$ for each $(R \times_T S)$-module M, there is an isomorphism of rings*

$$\overline{R \times_T S} \cong \overline{R} \times_{\overline{T}} \overline{S}$$

and there are exact sequences of $(\overline{R \times_T S})$-modules

$$0 \longrightarrow \overline{I} \oplus \overline{J} \longrightarrow \overline{R} \oplus \overline{S} \xrightarrow{\overline{\varepsilon_R} \oplus \overline{\varepsilon_S}} \overline{T} \oplus \overline{T} \longrightarrow 0 \tag{1.3.1}$$

$$0 \longrightarrow \overline{R \times_T S} \xrightarrow{\overline{\eta}} \overline{R} \oplus \overline{S} \xrightarrow{(\overline{\varepsilon_R}, -\overline{\varepsilon_S})} \overline{T} \longrightarrow 0 \tag{1.3.2}$$

The sequence $\boldsymbol{x}$ is $R \times_T S$-regular if and only if it is R-regular and S-regular.

Proof. One has $\mathrm{H}_n(\boldsymbol{x}, T) = 0$ for $n \geq 1$, so (1.1.1) induces an exact sequence of Koszul homology modules, which contains (1.3.1). It also gives an isomorphism

$$\mathrm{H}_1(\boldsymbol{x}, P) \cong \mathrm{H}_1(\boldsymbol{x}, R) \oplus \mathrm{H}_1(\boldsymbol{x}, S)\,,$$

which shows that $\boldsymbol{x}$ is P-regular if and only if it is R-regular and S-regular.

The exact sequence of Koszul homology modules induced by (1.1.2) contains the exact sequence (1.3.2), which, in turn implies the desired isomorphism of rings. $\square$

We relate numerical invariants of P to the corresponding ones of R, S, and T.

1.4. When Q is a local ring and N a finite Q-module, $\dim_Q N$ denotes its Krull dimension and $\operatorname{depth}_Q N$ its depth of N. Recall that if $P \to Q$ is a finite homomorphism of local rings, then one has $\dim_P N = \dim_Q N$ and $\operatorname{depth}_P N = \operatorname{depth}_Q N$.

We set $\dim Q = \dim_Q N$ and $\operatorname{depth} Q = \operatorname{depth}_Q Q$; thus, there are equalities $\dim Q = \dim_P Q$ and $\operatorname{depth} Q = \operatorname{depth}_P Q$.

Recall that $\operatorname{edim} Q$ denotes the *embedding dimension* of Q, defined to be the minimal number of generators of its maximal ideal.

Lemma 1.5. *The following (in)equalities hold:*

(1.5.1) $$\operatorname{edim}(R \times_T S) \geq \operatorname{edim} R + \operatorname{edim} S - \operatorname{edim} T\,.$$

(1.5.2) $$\dim(R \times_T S) = \max\{\dim R\,, \dim S\} \geq \min\{\dim R\,, \dim S\} \geq \dim T\,.$$

(1.5.3) $$\operatorname{depth}(R \times_T S) \geq \min\{\operatorname{depth} R\,, \operatorname{depth} S\,,\ \operatorname{depth} T + 1\}\,.$$

(1.5.4) $$\operatorname{depth} T \geq \min\{\operatorname{depth} R, \operatorname{depth} S, \operatorname{depth}(R \times_T S) - 1\}\,.$$

Proof. Lemma 1.2 gives an exact sequence of P-modules

$$0 \to \mathfrak{p} \to \mathfrak{r} \oplus \mathfrak{s} \to \mathfrak{t} \to 0$$

Tensoring it with $P/\mathfrak{p}$ over P, we get an exact sequence of k-vector spaces

$$\mathfrak{p}/\mathfrak{p}^2 \to \mathfrak{r}/\mathfrak{r}^2 \oplus \mathfrak{s}/\mathfrak{s}^2 \to \mathfrak{t}/\mathfrak{t}^2 \to 0$$

because we have $\mathfrak{p}\mathfrak{r} = \mathfrak{r}^2$, $\mathfrak{p}\mathfrak{s} = \mathfrak{s}^2$, and $\mathfrak{p}\mathfrak{t} = \mathfrak{t}^2$, due to the surjective homomorphisms $R \leftarrow P \to S \to T \leftarrow R$. These maps also give $\min\{\dim R\,, \dim S\} \geq \dim T$ and $\dim P \geq \max\{\dim R\,, \dim S\}$, while the inclusion η from (1.1.2) yields

$$\max\{\dim_P R\,, \dim_P S\} = \dim_P(R \oplus S) \geq \dim_P P\,.$$

For (1.5.3) and (1.5.4), apply the Depth Lemma, see [4, 1.2.9], to (1.1.2). □

For a local ring $(Q, \mathfrak{q}, k)$ and Q-module N, set $\operatorname{Soc} N = \{n \in N \mid \mathfrak{q}n = 0\}$. When $\boldsymbol{x}$ is a maximal N-regular sequence, $\operatorname{rank}_k \operatorname{Soc}(N/\boldsymbol{x}N)$ is a positive integer that does not depend on $\boldsymbol{x}$, see [4, 1.2.19], denoted $\operatorname{type}_Q N$. Set $\operatorname{type} Q = \operatorname{type}_Q Q$; thus, Q is Gorenstein if and only if it is Cohen-Macaulay and $\operatorname{type} Q = 1$.

We interpolate a useful general observation that uses fiber producs.

Lemma 1.6. *Let $(Q, \mathfrak{q}, k)$ be a local ring and W a k-subspace of $(\operatorname{Soc}(Q) + \mathfrak{q}^2)/\mathfrak{q}^2$.*

There exists a ring isomorphism $Q \cong B \times_k C$, where $(B, \mathfrak{b}, k)$ and $(C, \mathfrak{c}, k)$ are local rings, such that $\mathfrak{c}^2 = 0$ and $\mathfrak{c} \cong W$.

If $W = \operatorname{Soc}(Q) + \mathfrak{q}^2)/\mathfrak{q}^2$, then $\operatorname{Soc}(B) \subseteq \mathfrak{b}^2$.

Proof. When $\operatorname{Soc}(Q)$ is in $\mathfrak{q}^2$, set $B = Q$ and $C = k$. Else, pick in $\operatorname{Soc} Q$ a set $\boldsymbol{x}$ that maps bijectively to a basis of W, then choose in $\mathfrak{q}$ a set $\boldsymbol{y} \subset \mathfrak{q}$, so that $\boldsymbol{x} \cup \boldsymbol{y}$ maps bijectively to a basis of $\mathfrak{q}/\mathfrak{q}^2$. Set $B = Q/(\boldsymbol{x})$ and $C = Q/(\boldsymbol{y})$. One then has $\mathfrak{q} = (\boldsymbol{x}) + (\boldsymbol{y})$, hence $B \otimes_Q C \cong k$, and also $(\boldsymbol{x}) \cap (\boldsymbol{y}) = 0$, so $Q \cong B \times_k C$ by (1.0.3). The desired properties of B and C are verified by elementary calculations. □

The next two results concern ring-theoretic properties of fiber products.

Proposition 1.7. *Assume that T is Cohen-Macaulay, and set $d = \dim T$.*

The ring $R \times_T S$ is Cohen-Macaulay of dimension d if and only if R and S are.

When $R \times_T S$ is Cohen-Macaulay of dimension d the following inequalities hold:

$$\begin{aligned}\operatorname{type} R + \operatorname{type} S &\geq \operatorname{type}(R \times_T S)\\ &\geq \max\{\operatorname{type} R + \operatorname{type} S - \operatorname{type} T, \operatorname{type}_R I + \operatorname{type}_S J\}\,.\end{aligned}$$

If, in addition, I and J are non-zero, then $R \times_T S$ is not Gorenstein.

Proof. The first assertion follows directly from Lemmas 1.3 an 1.5, so assume that P is Cohen-Macaulay of dimension d. Choosing in P an $(P \oplus T)$-regular sequence of length d, from (1.3.2) we get an exact sequence of k-vector spaces

$$0 \longrightarrow \operatorname{Soc}(\overline{P}) \xrightarrow{\operatorname{Soc}\overline{\eta}} \operatorname{Soc}\overline{R} \oplus \operatorname{Soc}\overline{S} \xrightarrow{(\operatorname{Soc}\overline{\varepsilon_R}, -\operatorname{Soc}\overline{\varepsilon_S})} \operatorname{Soc}\overline{T}$$

It provides the inequalities involving $\operatorname{type} R$ and $\operatorname{type} S$. Formula (1.3.1) gives $\overline{\varepsilon_R}(\operatorname{Soc}\overline{I}) = 0 = \overline{\varepsilon_S}(\operatorname{Soc}\overline{J})$, so the sequence above yields $\overline{\eta}(\operatorname{Soc}\overline{P}) \supseteq \operatorname{Soc}\overline{I} \oplus \operatorname{Soc}\overline{J}$. When $I \neq 0 \neq J$ holds, we get $\overline{I} \neq 0 \neq \overline{J}$ by Nakayama's Lemma. Since $\overline{R}$ and $\overline{S}$ are artinian, one has $\operatorname{Soc}\overline{I} \neq 0 \neq \operatorname{Soc}\overline{J}$, whence $\operatorname{type} P \geq 2$. □

When $\varepsilon_R \colon R \to R/I$ is the canonical map and $\varepsilon_S = \varepsilon_R$, the ring $R \bowtie I = R \times_{R/I} R$ has been studied under the name *amalgamated duplication of R along I*. We complete and strengthen results of D'Anna and Shapiro:

Theorem 1.8. *Let R be a local ring, d its Krull dimension, and I a non-unit ideal.*

The ring $R \bowtie I$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and I is a maximal Cohen-Macaulay R-module.

The ring $R \bowtie I$ is Gorenstein if and only if R is Cohen-Macaulay and I is a canonical module for R, and then R/I is Cohen-Macaulay with $\dim(R/I) = d - 1$.

We start by listing those assertions in the theorem that are already known.

1.9. Assume that the ring R is Cohen-Macaulay.

1.9.1. If I is a maximal Cohen-Macaulay module, then $R \bowtie I$ is Cohen-Macaulay: This is proved by D'Anna in [6, Discussion 10].

1.9.2. If I is a canonical module for R, then $R \bowtie I$ is Gorenstein: This follows from a result of Eisenbud; see [6, Theorem 12].

1.9.3. If $R \bowtie I$ is Gorenstein *and I contains a regular element*, then I is a canonical module for R: In D'Anna's proof of [6, Theorem 11], this is deduced from [6, Proposition 3]; the italicized part of the hypothesis does not appear in the statement of that proposition, but Shapiro [22, 2.1] shows that it is needed.

1.9.4. If $R \bowtie I$ is Gorenstein and $\dim R = 1$, then I contains a regular element: This is proved by Shapiro, see [22, 2.4]; in the statement of that result it is also assumed that R reduced, but this hypothesis is not used in the proof.

Proof of Theorem 1.8. Set $P = R \bowtie I$ and $d = \dim R$; thus, $\dim P = d$ by (1.5.2).

We obtain the first assertion from a slight variation of the argument for 1.9.1. The map $R \to R \times R$, given by $r \mapsto (r, r)$, defines a homomorphisms of rings $R \to P$ that turns P into a finite R-module. Thus, P is a Cohen-Macaulay ring if and only if it is Cohen-Macaulay as an R-module; see 1.4. This module is isomorphic to $R \oplus I$, because each element $(r, s) \in P$ has a unique expression of the form $(r, r) + (0, s - r)$. It follows that P is Cohen-Macaulay if and only if R is Cohen-Macaulay and I is a maximal Cohen-Macaulay R-module.

In view of 1.9.2, for the rest of the proof we may assume P Gorenstein.

Set $T = R/I$. We have $\operatorname{depth} T \geq d - 1 \geq 0$ by (1.5.4) and Proposition 1.7. By the already proved assertion, R is Cohen-Macaulay with $\operatorname{depth} R = d$, so we can choose in P a T-regular and R-regular sequence $\boldsymbol{x}$ of length $d - 1$; for each P-module M set $\overline{M} = M/\boldsymbol{x}M$. By Lemma 1.3, $\boldsymbol{x}$ is P-regular, $\overline{I}$ is an ideal in $\overline{R}$ and there are isomorphisms of rings $\overline{T} \cong \overline{R}/\overline{I}$ and $\overline{P} \cong \overline{R} \bowtie \overline{I}$. As $\overline{R}$ is Cohen-Macaulay with $\dim \overline{R} = 1$ and $\overline{P}$ is Gorenstein, 1.9.4 shows that $\overline{I}$ contains an $\overline{R}$-regular element. This yield $\dim \overline{T} = 0$, hence $\dim T = d - 1$, so T is Cohen-Macaulay. Since R is Cohen-Macaulay as well, we have $\operatorname{grade}_R T = \dim R - \dim T = 1$, so I contains a regular element, and hence I is a canonical module for R, due to 1.9.3. □

2. Connected sums

A *connected sum diagram* of commutative rings is a commutative diagram

$$
\begin{tikzcd}[row sep=small]
 & R \arrow[dr, "\varepsilon_R"] & \\
V \arrow[ur, "\iota_R"] \arrow[dr, "\iota_S"'] & & T \\
 & S \arrow[ur, "\varepsilon_S"'] &
\end{tikzcd}
\tag{2.0.1}
$$

where V is a T-module, ι_R a homomorphism of R-modules (with R acting on V through ε_R) and ι_S a homomorphism of S-modules (with S acting on V via ε_S).

Evidently, $\{(\iota_R(v), \iota_S(v)) \in R \times S \mid v \in V\}$ is an ideal of $R \times_T S$. We define the *connected sum of R and S along the diagram* (2.0.1) to be the ring

$$R\#_T S = (R \times_T S)/\{(\iota_R(v), \iota_S(v)) \mid v \in V\}\,. \tag{2.0.2}$$

As in the case of fiber products, the maps in the diagram are suppressed from the notation, although the resulting ring does depend on them; see Example 3.1. The choices of name and notation are explained in Example 3.6.

We fix the setup and notation for this section as follows:

2.1. The rings in diagram (2.0.1) are local: $(R, \mathfrak{r}, k)$, $(S, \mathfrak{s}, k)$ and $(T, \mathfrak{t}, k)$.

The maps ε_R and ε_S are surjective; set $I = \operatorname{Ker}(\varepsilon_R)$, $J = \operatorname{Ker}(\varepsilon_S)$, also

$$P = R \times_T S \quad \text{and} \quad Q = R\#_T S\,.$$

The maps ι_R and ι_S are injective, so there are exact sequences of finite P-modules

$$0 \longrightarrow V \oplus V \xrightarrow{\iota_R \oplus \iota_S} R \oplus S \longrightarrow R/\iota_R(V) \oplus S/\iota_S(V) \longrightarrow 0 \tag{2.1.1}$$

$$0 \longrightarrow V \xrightarrow{\iota} R \times_T S \xrightarrow{\kappa} R\#_T S \longrightarrow 0 \tag{2.1.2}$$

where $\iota\colon v \mapsto (\iota_R(v), \iota_S(v))$ and κ is the canonical surjection.

A length count in (2.1.2), using formula (1.1.3), yields

$$\ell(R\#_T S) + \ell(T) + \ell(V) = \ell(R) + \ell(S)\,. \tag{2.1.3}$$

2.2. The ring Q is local and we write $(Q, \mathfrak{q}, k)$, unless $Q = 0$. The condition $Q = 0$ is equivalent to $\iota_R(V) = R$, and also to $\iota_S(V) = S$: This follows from the fact that $(P, \mathfrak{p}, k)$ is a local ring with $\mathfrak{p} = \mathfrak{r} \times_T \mathfrak{s}$, see Lemma 1.2.

When $I = 0$ one has $R \times_T S \cong S$, hence $R\#_T S \cong S/\iota_S(V)$.

Lemma 2.3. *If a sequence $\boldsymbol{x}$ in $R \times_T S$ is regular on $R/\iota_R(V)$, $S/\iota_S(V)$, T, and V, then it is also regular on R, S, $R \times_T S$, and $R\#_T S$, and there is an isomorphism*

$$\overline{R\#_T S} \cong \overline{R}\#_{\overline{T}}\overline{S}$$

of rings, where $\overline{M}$ denotes $M/\boldsymbol{x}M$ for every $R \times_T S$-module M.

Proof. The sequence (2.1.1) induces an exact sequence of Koszul homology modules

$$0 \longrightarrow \mathrm{H}_1(\boldsymbol{x}, R) \oplus \mathrm{H}_1(\boldsymbol{x}, S) \longrightarrow 0 \longrightarrow \overline{V} \oplus \overline{V} \xrightarrow{\overline{\iota_R} \oplus \overline{\iota_S}} \overline{R} \oplus \overline{S} \tag{2.3.1}$$

It follows that $\boldsymbol{x}$ is R-regular and S-regular. Lemma 1.3 shows that it is also P-regular, so (2.1.2) induces an exact sequence of Koszul homology modules

$$0 \longrightarrow \mathrm{H}_1(\boldsymbol{x}, Q) \longrightarrow \overline{V} \xrightarrow{\overline{\iota}} \overline{P} \xrightarrow{\overline{\kappa}} \overline{Q} \longrightarrow 0$$

Note that $\bar{\iota}$ is equal to the composition of the diagonal map $\overline{V} \to \overline{V} \oplus \overline{V}$ and $\overline{\iota_R} \oplus \overline{\iota_S} \colon \overline{V} \oplus \overline{V} \to \overline{R} \oplus \overline{S}$. Both are injective, the second one by (2.3.1), so $\bar{\iota}$ is injective as well. We get $\mathrm{H}_1(\boldsymbol{x}, Q) = 0$, so $\boldsymbol{x}$ is Q-regular. After identifying $\overline{P}$ and $\overline{R} \times_{\overline{T}} \overline{S}$ through Lemma 1.3, we get $\overline{Q} \cong \overline{R} \#_{\overline{T}} \overline{S}$ from the injectivity of $\bar{\iota}$. □

Proposition 2.4. *If the rings, $R/\iota_R(V)$, $S/\iota_S(V)$, T, and the T-module V are Cohen-Macaulay of dimension d, then so are the rings R, S, $R \times_T S$, and $R \#_T S$.*

Proof. The exact sequence (2.1.1) implies that R and S are Cohen-Macaulay of dimension d. Proposition 1.7 then shows that so is P; this gives $\dim Q \leq d$. Let $\boldsymbol{x}$ be a sequence of length d in P, which is regular on $(R/\iota_R(V) \oplus S/\iota_S(V) \oplus T \oplus V)$. By Lemma 2.3, it is also Q-regular, so Q is Cohen-Macaulay of dimension d. □

To describe those situations, where connected sums do not produce new rings, we review basic properties of Hilbert-Samuel multiplicities.

2.5. Let $(P, \mathfrak{p}, k)$ be a Cohen-Macaulay local ring of dimension d.

When k is infinite, the *multiplicity* $e(P)$ can be expressed as

$$e(P) = \inf\{\ell(P/\boldsymbol{x}P) \mid \boldsymbol{x} \text{ is a } P\text{-regular sequence in } P\}\,;$$

see [4, 4.7.11]. If $P \to P'$ is a surjective homomorphism of rings, and P' is Cohen-Macaulay of dimension d, then by [20, Ch. 1, 3.3] there exists in P a sequence $\boldsymbol{x}$ that is both P-regular and P'-regular, and $e(P') = \ell(P'/\boldsymbol{x}P')$ holds.

When k is finite, one has $e_P(M) = e_{P[y]_{\mathfrak{p}[y]}}\big(M \otimes_P P[y]_{\mathfrak{p}[y]}\big)$.

The ring P is regular if and only if if $e(P) = 1$.

It is a quadratic hypersurface if and only if $e(P) = 2$.

Proposition 2.6. *Assume that the rings $R/\iota_R(V)$, $S/\iota_S(V)$, and T, and the T-module V, are Cohen-Macaulay, and their dimensions are equal.*

When R is regular one has $I = 0$ and $R \#_T S \cong S/\iota_S(V)$.

When R is a quadratic hypersurface and $I \neq 0$, one has $R \#_T S \cong S$.

Proof. Set $d = \dim T$. By Proposition 2.4, P, Q, R, and S are Cohen-Macaulay of dimension d. Thus, every P-regular sequence is also regular on Q, R, and S.

When R is regular it is a domain; $\dim R = \dim T$ implies $I = 0$, so 2.2 applies.

Assume $I \neq 0$ and $e(R) = 2$. Tensoring, if necessary, the diagram (2.0.1) with $P[x]_{\mathfrak{p}[x]}$ over P, we may assume that k is infinite. By 2.5, there is a P- and R-regular sequence $\boldsymbol{x}$ of length d in P, such that $\ell(\overline{R}) = 2$, where overbars denote reduction modulo $\boldsymbol{x}$. From (1.3.1) and $I \neq 0$ one gets $\ell(\overline{T}) = \ell(\overline{R}) - \ell(\overline{I}) \leq 1$. This implies $\ell(\overline{T}) = 1 = \ell(\overline{V})$, so Lemma 2.3 and (2.1.3) give $\ell(\overline{Q}) = \ell(\overline{S})$.

Setting $K = \operatorname{Ker}(Q \to S)$, one sees that the induced sequence

$$0 \longrightarrow \overline{K} \longrightarrow \overline{Q} \longrightarrow \overline{S} \longrightarrow 0$$

is exact, due to the S-regularity of $\boldsymbol{x}$, hence $\overline{K} = 0$, and thus $K = 0$. □

A construction of canonical modules sets the stage for the next result.

2.7. The ideal $(0 : I)$ of R is a T-module, which is isomorphic to $\operatorname{Hom}_R(T, R)$. Similarly, $(0 : J) \cong \operatorname{Hom}_S(T, S)$ as T-modules. If R and S are Gorenstein, T is Cohen-Macaulay, and all three rings have dimension d, then $(0 : I)$ and $(0 : J)$ are isomorphic T-modules, since both are canonical modules for T; see [4, 3.3.7].

Theorem 2.8. *Let R and S be Gorenstein local rings of dimension d, let T be a Cohen-Macaulay local ring of dimension d and V a canonical module for T.*

Let ε_R, ε_S, ι_R, and ι_S be maps that satisfy the conditions in 2.1 and, in addition,

$$\iota_R(V) = (0 : I) \quad \textit{and} \quad \iota_S(V) = (0 : J)\,.$$

If $I \neq 0$ or $J \neq 0$, then $R\#_T S$ is a Gorenstein local ring of dimension d.

Remark. The condition $I \neq 0$ is equivalent to $J \neq 0$.

Indeed $I = 0$ implies $R = (0 : I) = \iota_R(V)$, hence $\varepsilon_S\iota_S(V) = \varepsilon_R\iota_R(V) = T$. In particular, for some $v \in V$ one has $\varepsilon_S\iota_S(v) = 1 \in T$, hence $S = S\iota_S(v) \subseteq (0 : J)$, and thus $J = 0$. By symmetry, $J = 0$ implies $I = 0$.

Proof of Theorem 2.8. The T-module V is Cohen-Macaulay of dimension d, see [4, 3.3.13]. The rings $R/(0 : I)$ and $S/(0 : J)$ have the same property, by [17, 1.3]. Proposition 2.4 now shows that the ring Q is Cohen-Macaulay of dimension d.

Choose in P an $(R/\iota_R(V)\oplus S/\iota_S(V)\oplus Q\oplus T\oplus V)$-regular sequence $\boldsymbol{x}$ of length d. It suffices to show that $Q/\boldsymbol{x}Q$ is Gorenstein. The $T/\boldsymbol{x}T$-module $V/\boldsymbol{x}V$ is canonical, see [4, 3.3.5], so reduction modulo $\boldsymbol{x}$ preserves the hypotheses of the theorem. In view of Lemma 2.3, we may assume that all rings involved are artinian.

Now we have $\operatorname{Soc} V = Tu$ for some $u \in \operatorname{Soc} T$; see [4, 3.3.13]. To prove that Q is Gorenstein we show that $\kappa(\iota_R(u), 0)$ generates $\operatorname{Soc} Q$. Write $q \in \operatorname{Soc} Q$ in the form

$$q = \kappa(a, b) \quad \text{with} \quad (a, b) \in \mathfrak{r} \times_T \mathfrak{s} = \mathfrak{p}\,.$$

As S is Gorenstein, one has $\iota_S(u) \in \operatorname{Soc} S \subseteq J$. For every $i \in I$ this gives $\varepsilon_R(i) = 0 = \varepsilon_S\iota_S(u)$. Thus, $(i, \iota_S(u))$ is in $\mathfrak{p}$, so $\kappa(i, \iota_S(u)) \cdot q \in \mathfrak{q} \cdot q = 0$ holds, hence

$$(ia, 0) = (i, \iota_S(u)) \cdot (a, b) = (\iota_R(x), \iota_S(x))$$

for some $x \in V$. Since ι_S is injective we get $x = 0$, hence $ia = 0$. As i was arbitrarily chosen in I, this implies $a \in (0 : I)$; that is, $a = \iota_R(v)$ for some v in V. By symmetry, we conclude $b = \iota_S(w)$ for some $w \in V$. As a consequence, we get

$$q = \kappa(\iota_R(v), \iota_S(w)) \quad \text{with} \quad v, w \in V\,.$$

Pick any t in $\mathfrak{t}$, then choose r in $\mathfrak{r}$ and s in $\mathfrak{s}$ with $\varepsilon_R(r) = t = \varepsilon_S(s)$. Thus, (r, s) is in $\mathfrak{p}$, hence $\kappa(r, s)$ is in $\mathfrak{q}$, whence $\kappa(r, s) \cdot q = 0$. We then have

$$(\iota_R(tv), \iota_S(tw)) = (r\iota_R(v), s\iota_S(w)) = (r, s) \cdot (\iota_R(v), \iota_S(w)) = (\iota_R(y), \iota_S(y))$$

for some $y \in V$. This yields $\iota_R(tv) = \iota_R(y)$ and $\iota_S(tw) = \iota_S(y)$, hence $tv = y = tw$, due to the injectivity of ι_R and ι_S; in other words, $t(v - w) = 0$. Since t was an arbitrary element of $\mathfrak{t}$, we get $\mathfrak{t}(v - w) = 0$, hence $v = w + t'u$ for some $t' \in T$. Choosing r' in R and s' in S with $\varepsilon_R(r') = t' = \varepsilon_S(s')$, we have $(r', s') \in P$ and

$$\begin{aligned} q &= \kappa(\iota_R(w), \iota_S(w)) + \kappa(\iota_R(t'u), 0) \\ &= \kappa\big((r', s') \cdot (\iota_R(u), 0)\big) \\ &= \kappa(r', s') \cdot \kappa(\iota_R(u), 0) \end{aligned}$$

As q can be any element of $\operatorname{Soc} Q$, we get $\operatorname{Soc} Q = Q \cdot \kappa(\iota_R(u), 0)$, as desired. $\square$

3. Examples and variations

We collect examples to illustrate the hypotheses and the conclusions of results proved above, and review variants and antecedents of the notion of connected sum.

Seemingly minor perturbations of diagram (2.0.1) may lead to non-isomorphic connected sum rings. Next we produce a concrete illustration. See also Example 3.7 for connected sums that are not isomorphic *as graded algebras*.

Example 3.1. Over the field $\mathbb{Q}$ of rational numbers, form the algebras

$$R = \mathbb{Q}[x]/(x^3)\,, \quad S = \mathbb{Q}[y]/(y^3)\,, \quad \text{and} \quad T = \mathbb{Q}\,.$$

Letting both $\varepsilon_R\colon R \to T$ and $\varepsilon_S\colon S \to T$ be the canonical surjections, one gets

$$R \times_T S = \mathbb{Q}[x,y]/(x^3, xy, y^3)\,.$$

Set $V = \mathbb{Q}$ and let $\iota_R\colon V \to R$ and $\iota_S\colon V \to S$ be the maps $q \mapsto qx$ and $q \mapsto qy$, respectively. The connected sum defined by these data is a local ring $(Q, \mathfrak{q}, k)$ with

$$Q = \mathbb{Q}[x,y]/(x^2 - y^2, xy)\,.$$

On the other hand, take the same maps ε_R, ε_S, and ι_S as above, and replace ι_R with the map $q \mapsto pq$, where p is a prime number that is not congruent to 3 modulo 4. We then get as connected sum a local ring $(Q', \mathfrak{q}', \mathbb{Q})$ with

$$Q' = \mathbb{Q}[x', y']/(x'^2 - py'^2, x'y')\,.$$

We claim that these rings are not isomorphic. In fact, more is true:

Every ring homomorphism $\kappa\colon Q' \to Q$ satisfies $\kappa(\mathfrak{q}') \subseteq \mathfrak{q}^2$.

Indeed, any ring homomorphism of $\mathbb{Q}$-algebras is $\mathbb{Q}$-linear, so κ is a homomorphism of $\mathbb{Q}$-algebras. The images of x' and y' can be written in the form

$$\kappa(x') = ax + by + cy^2$$
$$\kappa(y') = dx + ey + fy^2$$

for appropriate rational numbers a, b, c, d, e, and f. In Q this gives equalities

$$(a^2+b^2)x^2 = a^2x^2 + b^2y^2 = \kappa(x'^2) = \kappa(py'^2) = p(d^2x^2 + e^2y^2) = p(d^2+e^2)x^2$$

We need to show that the only rational solution of the equation

$$a^2 + b^2 = p(d^2 + e^2) \tag{3.1.1}$$

is the trivial one. If not, then $a^2 + b^2 \neq 0$. Clearing denominators, we may assume $a, b, c, d \in \mathbb{Z}$ and write $a^2 + b^2 = p^i g$ and $d^2 + e^2 = p^j h$ with integers $g, h, i, j \geq 0$, such that gh is not divisible by p. By Fermat's Theorem on sums of two squares, see [21, §5.6], i and j must be even. This is impossible, as (3.1.1) forces $i = j + 1$.

Now we turn to graded rings and degree-preserving homomorphisms.

Recall that the *Hilbert series* of a graded vector space D over a field k, with $\operatorname{rank}_k D_n < \infty$ for all $n \in \mathbb{Z}$ and $D_n = 0$ for $n \ll 0$, is the formal Laurent series

$$H_D = \sum_{n > -\infty} \operatorname{rank}_k(D_n) z^n \in \mathbb{Z}[\![z]\!][z^{-1}]\,.$$

Remark 3.2. Let k be a field and assume that the rings R and S in diagram (1.0.1) are commutative finitely generated $\mathbb{N}$-graded k-algebras with $R_0 = k = S_0$, and the maps are homogeneous. Equation (1.1.3) then can be refined to:

$$H_{R \times_T S} = H_R + H_S - H_T\,, \tag{3.2.1}$$

and the obvious version of Theorem 1.7 for graded rings holds as well.

Assume, in addition, that in the diagram (2.0.1) all maps are homogeneous. Equation (2.1.3) then can be refined to:

$$H_{R\#_T S} = H_R + H_S - H_T - H_V \,. \tag{3.2.2}$$

Diligence is needed to state a graded analog of Theorem 2.8. Recall that for each finite graded T-module N one has $H_M = h_N/g_N$ with $h_N \in \mathbb{Z}[z^{\pm 1}]$ and $g_N \in \mathbb{Z}[z]$, and that the integer $a(N) = \deg(h_N) - \deg(g_N)$ is known as the *a-invariant* of N.

Theorem 3.3. *Let $R \xrightarrow{\varepsilon_R} T \xleftarrow{\varepsilon_S} S$ be surjective homomorphisms of commutative $\mathbb{N}$-graded k-algebras of dimension d, with $R_0 = k = S_0$. Assume that R and S are Gorenstein, T is Cohen-Macaulay, and V is a canonical module for T.*

A connected sum diagram (2.0.1)*, with ι_R and ι_S isomorphisms of graded modules, exists if and only if $a(R) = a(S)$. When this is the case the graded algebra $R\#_T S$ is Gorenstein of dimension d, with $a(R\#_T S) = a(R)$ and*

$$H_{R\#_T S}(z) = H_R(z) + H_S(z) - H_T(z) - (-1)^d z^{a(R)} \cdot H_T(z^{-1}) \,. \tag{3.3.1}$$

Proof. From [4, 4.4.5] one obtains

$$H_{\operatorname{Hom}_R(T,R)}(z) = z^{a(R)} \cdot H_{\operatorname{Hom}_R(T,R(a))}(z) = (-1)^d z^{a(R)} \cdot H_T(z^{-1}) \,,$$

and a similar formula with S in place of R. Thus, $\operatorname{Hom}_R(T,R) \cong \operatorname{Hom}_S(T,S)$ holds as *graded* T-modules if and only if $a(R) = a(S)$. In this case, Theorem 2.8 (or its proof) shows that $R\#_T S$ is Gorenstein, and formula (3.2.2) yields (3.3.1). □

Generation in degree 1 does not transfer from R and S to $R \times_T S$ or $R\#_T S$:

Example 3.4. Set $T = k[z]/(z^2)$ and form the homomorphisms of k-algebras

$$R = k[x]/(x^5) \to T \leftarrow k[y]/(y^5) = S \quad \text{with} \quad x \mapsto z \leftarrow y \,.$$

Choose $V = T$ and define homomorphisms $R \xleftarrow{\iota_R} V \xrightarrow{\iota_S} S$ by setting $\iota_R(1) = x^3$ and $\iota_S(1) = y^3$. The graded k-vector space $R \times_T S$ has a homogeneous basis

$$\{(x^i, y^i)\}_{0\le i\le 4} \cup \{(0, y^j)\}_{2\le j\le 4} \,,$$

which yields $R \times_T S \cong k[u,v]/(u^5, uv^2, v^2 - u^2 v)$ with $\deg(u) = 1$ and $\deg(v) = 2$.

The canonical module of T is isomorphic to $(x^3) \subset R$ and $(y^3) \subset S$. Therefore, one gets $R\#_T S \cong k[u,v]/(v^2 - u^2 v, 2uv - u^3)$, with degrees as above.

Remark 3.5. For the definition of connected sum given in (2.0.2) to work in a non-commutative context, the only change needed is to require that the maps ι_R and ι_S in diagram (2.0.1) be homomorphisms of T-*bimodules.*

When the maps in the diagram are homogeneous homomorphisms of rings and bimodules, the resulting connected sum is a graded ring.

Remark 3.5 is used implicitly in the next two examples, which deal with graded-commutative, rather than commutative, k-algebras.

Example 3.6. Let M and N be compact connected oriented smooth manifolds of the same dimension, say n. The connected sum $M\#N$ is the manifold obtained by removing an open n-disc from each manifold and gluing the resulting manifolds with boundaries along their boundary spheres through an orientation-reversing homeomorphism. The cohomology algebras with coefficients in a field k satisfy $\mathrm{H}^*(M\#N) \cong \mathrm{H}^*(M)\#_k \mathrm{H}^*(N)$, with $\varepsilon_{\mathrm{H}^*(M)}$ and $\varepsilon_{\mathrm{H}^*(N)}$ the canonical augmentations, $V = k$, and $\iota_{\mathrm{H}^*(M)}(1)$ and $\iota_{\mathrm{H}^*(N)}(1)$ the orientation classes.

What may be the earliest discussion of connected sums in a ring-theoretical context followed very closely the topological model:

Example 3.7. Sah [19] formed connected sums of *graded Poincaré duality algebras* along their orientation classes, largely motivated by the following special case:

A Poincaré duality algebra R with $R_i = 0$ for $i \neq 0, 1, 2$ is completely described by the quadratic form $R_1 \to k$, obtained by composing the map $x \mapsto x^2$ with the inverse of the orientation isomorphism $k \xrightarrow{\cong} R_2$. Such algebras are isomorphic if and only if the corresponding forms are equivalent, and the connected sum of two such algebras corresponds to the Witt sum of the corresponding quadratic forms.

4. Gorenstein colength

Let $(Q, \mathfrak{q}, k)$ be an artinian local ring and E an injective hull of k.

The *Gorenstein colength* of Q is defined in [1] to be the number

$$\operatorname{gcl} Q = \min\left\{ \ell(A) - \ell(Q) \;\middle|\; \begin{array}{c} Q \cong A/I \text{ with } A \text{ an artinian} \\ \text{Gorenstein local ring} \end{array} \right\}.$$

One has $\operatorname{gcl} Q = 0$ if and only if Q is Gorenstein, and

$$0 \leq \operatorname{gcl} Q \leq \ell(Q) < \infty\,,$$

as the trivial extension $Q \ltimes E$ is Gorenstein, see [4, 3.3.6], and $\ell(Q \ltimes E) = 2\,\ell(Q)$.

Lemma 4.1. *If Q is a non-Gorenstein artinian local ring and $Q \to C$ is a surjective homomorphism with C Gorenstein, then the following inequality holds:*

$$\operatorname{gcl} Q \geq \operatorname{edim}(Q) - (\ell(Q) - \ell(C))\,.$$

Proof. Let $A \to Q$ be a surjection with $(A, \mathfrak{a}, k)$ Gorenstein and $\ell(A) - \ell(Q) = \operatorname{gcl} Q$. It factors through $\overline{A} \to Q$, where $\overline{A} = A/\operatorname{Soc} A$. Applying $\operatorname{Hom}_A(-, A)$ to the exact sequence $0 \to \mathfrak{a}/\mathfrak{a}^2 \to A/\mathfrak{a}^2 \to A/\mathfrak{a} \to 0$, one gets an exact sequence

$$0 \to (0 : \mathfrak{a})_A \to (0 : \mathfrak{a}^2)_A \to \operatorname{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A) \to 0$$

that yields $\operatorname{Soc}(\overline{A}) \cong \operatorname{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A)$. As $\mathfrak{a}$ annihilates $\mathfrak{a}/\mathfrak{a}^2$, the second module is isomorphic to $\operatorname{Hom}_k(\mathfrak{a}/\mathfrak{a}^2, k)$. Set $K = \operatorname{Ker}(\overline{A} \to C)$. Since $\ell(\operatorname{Soc}(C)) = 1$, the inclusion $\operatorname{Soc}(\overline{A})/(K \cap (\operatorname{Soc}(\overline{A})) \subseteq \operatorname{Soc}(C)$ gives the second inequality below:

$$\ell(K) \geq \ell(K \cap \operatorname{Soc} \overline{A}) \geq \ell(\operatorname{Soc}(\overline{A})) - 1 = \operatorname{edim} A - 1 \geq \operatorname{edim} Q - 1.$$

The desired inequality now follows from a straightforward length count:

$$\ell(A) - \ell(Q) = (\ell(K) + 1) - (\ell(Q) - \ell(C)) \geq \operatorname{edim} Q - (\ell(Q) - \ell(C))\,. \qquad \square$$

Rings of embedding dimension 1 need separate consideration.

4.2. Let $(S, \mathfrak{s}, k)$ be an artinian local ring with $\operatorname{edim} S \leq 1$.

The ring S is Gorenstein, and one has $S \cong C/(x^n)$ with $(C, (x), k)$ a discrete valuation ring and $n = \ell(S)$; thus, there is a surjective, but not bijective, homomorphism $B \to S$, where $B = C/(x^{n+1})$ is artinian, Gorenstein, with $\ell(B) = \ell(S) + 1$.

Proposition 4.3. *Let $(R, \mathfrak{r}, k)$ and $(S, \mathfrak{s}, k)$ be artinian local rings, with $\mathfrak{r} \neq 0 \neq \mathfrak{s}$.*

(1) *When R and S are Gorenstein, there is an inequality*

$$\operatorname{gcl}(R \times_k S) \geq \operatorname{edim} R + \operatorname{edim} S - 1\,;$$

equality holds if $\operatorname{edim} R = 1 = \operatorname{edim} S$.

(2) *When R is not Gorenstein, there are inequalities*

$$1 \le \operatorname{gcl}(R \times_k S) \le \begin{cases} \operatorname{gcl} R & \text{if} \quad \operatorname{edim} S = 1\,; \\ \operatorname{gcl} R + \operatorname{gcl} S - 1 & \text{if} \quad \operatorname{gcl} S \ge 1\,. \end{cases}$$

Proof. The ring $R \times_k S$ is not Gorenstein by Proposition 1.7, hence $\operatorname{gcl}(R \times_k S) \ge 1$.

(1) The ring $R \#_k S$ is Gorenstein by Theorem 2.8, so apply Lemma 4.1 to the homomorphism $R \times_k S \to R \#_k S$ and use $\operatorname{edim}(R \times_k S) = \operatorname{edim} R + \operatorname{edim} S$.

(2) Choose a surjective homomorphism $A \to R$ with A artinian Gorenstein and $\ell(A) = \ell(R) + \operatorname{gcl} R$. If $\operatorname{gcl} S \ge 1$, let $B \to S$ be a surjective homomorphism with B artinian Gorenstein and $\ell(B) = \ell(S) + \operatorname{gcl} S$; if $\operatorname{edim} S = 1$, let $B \to S$ be the map described in 4.2. In both cases there is a commutative diagram

$$\begin{tikzcd} & A \arrow[r, two heads] & R \arrow[dr, two heads] & \\ k \arrow[ur] \arrow[dr] & & & k \\ & B \arrow[r, two heads] & S \arrow[ur, two heads] & \end{tikzcd}$$

where two-headed arrows denote surjective homomorphisms of local rings, and the maps from k are isomorphisms onto the socles of A and B. Both compositions $R \leftarrow k \to S$ are zero, so there is a surjective homomorphism $A \#_k B \to R \times_k S$.

In the following string the inequality holds because $A \#_k B$ is Gorenstein, see Theorem 2.8, and the first equality comes from formulas (1.1.3) and (2.1.3):

$$\begin{aligned} \operatorname{gcl}(R \times_k S) &\le \ell(A \#_k B) - \ell(R \times_k S) \\ &= (\ell(A) + \ell(B) - 2) - (\ell(R) + \ell(S) - 1) \\ &= \operatorname{gcl} R + (\ell(B) - \ell(S) - 1) \end{aligned}$$

The desired upper bounds now follow from the choice of B. $\square$

As a first application, we give a new, simple proof of a result of Teter, [23, 2.2].

Corollary 4.4. *A local ring $(Q, \mathfrak{q}, k)$ with $\mathfrak{q}^2 = 0$ has $\operatorname{gcl} Q = 1$ or $\operatorname{edim} Q \le 1$.*

Proof. The condition $\mathfrak{q}^2 = 0$ is equivalent to $\mathfrak{q} = \operatorname{Soc} Q$. Set $\operatorname{rank}_k \mathfrak{q} = s$.

One has $s = \operatorname{edim} Q$, so we assume $s \ge 2$; we then have $\operatorname{gcl} Q \ge 1$. Lemma 1.6 gives $Q \cong R \times_k S$ where $(R, \mathfrak{r}, k)$ and $(S, \mathfrak{s}, k)$ are local rings, $\mathfrak{r}^2 = 0$, $\operatorname{edim} R = s - 1$, and $\operatorname{edim} S = 1$. If $s = 2$, then $\operatorname{edim} R = 1$, hence $\operatorname{gcl} Q = 1$ by Proposition 4.3(1). If $s \ge 3$, then $\operatorname{gcl} R = 1$ holds by induction, so Proposition 4.3(2) yields $\operatorname{gcl} Q = 1$. $\square$

Note that the conditions $\operatorname{gcl} Q = 1$ and $\operatorname{edim} Q \le 1$ are mutually exclusive; one or the other holds if and only if R is isomorphic to the quotient of some artinian Gorenstein ring by its socle, see 4.2. Such rings are characterized as follows:

4.5. Let $(Q, \mathfrak{q}, k)$ be an artinian local ring and E an injective envelope of k.

Teter [23, 2.3, 1.1] proved that there exists an isomorphism $Q \cong A/\operatorname{Soc}(A)$, with $(A, \mathfrak{a}, k)$ an artinian Gorenstein local ring, if and only if there is a homomorphism of Q-modules $\varphi\colon \mathfrak{q} \to \operatorname{Hom}_Q(\mathfrak{q}, E)$ satisfying $\varphi(x)(y) = \varphi(y)(x)$ for all $x, y \in \mathfrak{q}$.

His analysis includes the following observation: $E \cong \operatorname{Hom}_A(Q, A)$, so the exact sequence $0 \to k \to A \to Q \to 0$ induces an exact sequence $0 \to E \to A \to k \to 0$. It yields $E \cong \mathfrak{a}$, and thus a composed Q-linear surjection $E \cong \mathfrak{a} \to \mathfrak{a}/\operatorname{Soc}(A) = \mathfrak{q}$.

Using Teter's result, Huneke and Vraciu proved a partial converse:

4.6. If $\operatorname{Soc} Q \subseteq \mathfrak{q}^2$, 2 is invertible in Q, and there exists an epimorphism $E \to \mathfrak{q}$, then $Q \cong A/\operatorname{Soc}(A)$ with A Gorenstein; see [10, 2.5].

We lift the restriction on the socle of Q.

Theorem 4.7. *Let $(Q, \mathfrak{q}, k)$ be an artinian local ring, in which 2 is invertible, and let E be an injective hull of k.*

If there is an epimorphism $E \to \mathfrak{q}$, then $Q \cong A/\operatorname{Soc}(A)$ with A Gorenstein.

Proof. By Lemma 1.6, there is an isomorphism $Q \cong R \times_k S$, where $(R, \mathfrak{r}, k)$ is a local ring with $\operatorname{Soc}(R) \subseteq \mathfrak{r}^2$ and $(S, \mathfrak{s}, k)$ is a local ring with $\mathfrak{s}^2 = 0$. Choose a surjective homomorphism $P \to Q$ with P Gorenstein and set $E_R = \operatorname{Hom}_P(R, P)$ and $E_S = \operatorname{Hom}_P(S, P)$. We then have $E \cong \operatorname{Hom}_P(Q, P)$ and surjective homomorphisms

$$E_R \oplus E_S \xrightarrow{\alpha} E \xrightarrow{\beta} \mathfrak{q} = \mathfrak{r} \oplus \mathfrak{s} \xrightarrow{\gamma} \mathfrak{r}$$

where α is induced by the composition $Q \cong R \times_k S \hookrightarrow R \oplus S$, β comes from the hypothesis, and γ is the canonical map. Note that $\ell(E) = \ell(Q) > \ell(\mathfrak{q}) = \ell(\beta(E))$ implies $\operatorname{Ker}(\beta) \neq 0$; since $\ell(\operatorname{Soc} E) = 1$, we get $\operatorname{Soc} E \subseteq \operatorname{Ker}(\beta)$.

One has $\mathfrak{q}^2 \alpha(E_S) = \alpha(\mathfrak{q}^2 E_S) = 0$. This gives $\mathfrak{q}\alpha(E_S) \subseteq \operatorname{Soc}(E) \subseteq \operatorname{Ker}(\beta)$, hence $\mathfrak{q}\beta\alpha(E_S) = \beta(\mathfrak{q}\alpha(E_S)) = 0$, and thus $\beta\alpha(E_S) \subseteq \operatorname{Soc} \mathfrak{q}$. From here we get

$$\gamma\beta\alpha(E_S) \subseteq \gamma(\operatorname{Soc} \mathfrak{q}) \subseteq \operatorname{Soc} \mathfrak{r} \subseteq \operatorname{Soc} R \subseteq \mathfrak{r}^2 .$$

Using the inclusions above, we obtain a new string:

$$\mathfrak{r} = \gamma\beta\alpha(E_R \oplus E_S) = \gamma\beta\alpha(E_R) + \gamma\beta\alpha(E_S) \subseteq \gamma\beta\alpha(E_R) + \mathfrak{r}^2 .$$

By Nakayama's Lemma, $\gamma\beta\alpha$ restricts to a surjective homomorphism $E_R \to \mathfrak{r}$.

As E_R is an injective envelope of k over R, and $\operatorname{Soc} R$ is contained in $\mathfrak{r}^2$, we get $\operatorname{gcl} R = 1$ or $\operatorname{edim} R \leq 1$ from Huneke and Vraciu's theorem; see 4.6. On the other hand, we know from Lemma 4.4 that S satisfies $\operatorname{gcl} S = 1$ or $\operatorname{edim} S \leq 1$, so from Proposition 4.3 we conclude that $\operatorname{gcl} Q = 1$ or $\operatorname{edim} Q \leq 1$ holds. □

Finally, we take a look at the values of $\ell(A) - \ell(Q)$, when Q is fixed.

Remark 4.8. Let Q be an artinian local ring; set $\operatorname{edim} Q = e$ and $\operatorname{gcl} Q = g$.

If $e \leq 1$ or $g \geq 1$, then for every $n \geq 0$ there is an isomorphism $Q \cong A/I$, with A a Gorenstein local ring and $\ell(A) - \ell(Q) = g + n$.

Indeed, the case of $e = 1$ is clear from 4.2, so we assume $e \geq 2$. When $g \geq 1$, let $R \to Q$ be a surjective homomorphism with R Gorenstein and $\ell(R) = g$. For $S = k[x]/(x^{n+2})$, the canonical surjection $R \times_k S \to R \times_k k \cong R$ maps $\operatorname{Soc}(R) \oplus \operatorname{Soc}(S)$ to zero, and so factors through $R \#_k S$. Theorem 2.8 shows that this ring is Gorenstein, and formula (2.1.3) yields $\ell(R \#_k S) = g + (n + 2) - 2$.

5. Cohomology algebras

Our next goal is to compute the cohomology algebra of a connected sum of artinian Gorenstein rings over their common residue field, in terms of the cohomology algebra of the original rings. The computation takes up three consecutive sections.

In this section we describe some functorial structures on cohomology.

5.1. Let $(P, \mathfrak{p}, k)$ be a local ring and $\kappa \colon P \to Q$ is a surjective ring homomorphism.

Let F be a minimal free resolution of k over P. One then has

$$\operatorname{Ext}_P^*(k, k) = \operatorname{Hom}_P(F, k) \quad \text{and} \quad \operatorname{Tor}^P_*(k, k) = F \otimes_P k .$$

Homological products turns $\operatorname{Tor}^P_*(k,k)$ into a graded-commutative algebra with divided powers, see [9, 2.3.5] or [3, 6.3.5]; this structure is preserved by the map

$$\operatorname{Tor}^\kappa_*(k,k)\colon\ \operatorname{Tor}^P_*(k,k) \to \operatorname{Tor}^Q_*(k,k)\,.$$

Composition products turn $\operatorname{Ext}^*_P(k,k)$ into a graded k-algebra see [9, Ch. II, §3], and the homomorphism of rings κ induces a homomorphism of graded k-algebras

$$\operatorname{Ext}^*_\kappa(k,k)\colon\ \operatorname{Ext}^*_Q(k,k) \to \operatorname{Ext}^*_P(k,k)\,.$$

For each $n\in\mathbb{Z}$, the canonical bilinear pairing

$$\operatorname{Ext}^n_P(k,k)\times\operatorname{Tor}^P_n(k,k)\to k$$

given by evaluation is non-degenerate; we use it to identify the graded vector spaces

$$\operatorname{Ext}^*_P(k,k)=\operatorname{Hom}_k(\operatorname{Tor}^P_*(k,k),k)\,.$$

Let $\pi^*(P)$ be the graded k-subspace of $\operatorname{Ext}^*_P(k,k)$, consisting of those elements that vanish on all products of elements in $\operatorname{Tor}^P_+(k,k)$ and on all divided powers $t^{(i)}$ of elements $t\in\operatorname{Tor}^P_{2j}(k,k)$ with $i\ge 2$ and $j\ge 1$. As $\pi^*(P)$ is closed under graded commutators in $\operatorname{Ext}^*_P(k,k)$, it is a graded Lie algebra, called the *homotopy Lie algebra* of P. The canonical map from the universal enveloping algebra of $\pi^*(P)$ to $\operatorname{Ext}^*_P(k,k)$ is an isomorphism; see [3, 10.2.1]. The properties of $\operatorname{Tor}^\kappa_*(k,k)$ and $\operatorname{Ext}^*_\kappa(k,k)$ show that κ induces a homomorphism of graded Lie algebras

$$\pi^*(\kappa)\colon \pi^*(Q)\to\pi^*(P)\,.$$

The maps $\operatorname{Tor}^\kappa_*(k,k)$, $\operatorname{Ext}^*_\kappa(k,k)$, and $\pi^*(\kappa)$ are functorial.

The next lemma can be deduced from [2, 3.3]. We provide a direct proof.

Lemma 5.2. *Given a local ring $(P,\mathfrak{p},k)$ and an exact sequence of P-modules*

$$0\longrightarrow V\xrightarrow{\ \iota\ }P\xrightarrow{\ \kappa\ }Q\longrightarrow 0$$

there is a natural exact sequence of k-vector spaces

$$0\longrightarrow\pi^1(Q)\xrightarrow{\pi^1(\kappa)}\pi^1(P)\longrightarrow\operatorname{Hom}_P(V,k)\xrightarrow{\ \widetilde{\iota}\ }\pi^2(Q)\xrightarrow{\pi^2(\kappa)}\pi^2(P)$$

Proof. The classical change of rings spectral sequence

$$\mathrm{E}_2^{p,q}=\operatorname{Ext}^p_Q(k,\operatorname{Ext}^q_P(Q,k))\ \underset{p}{\Longrightarrow}\ \operatorname{Ext}^{p+q}_P(k,k),$$

see [5, XVI.5.(2)$_4$], yields a natural exact sequence of terms of low degree

$$\begin{gathered}0\longrightarrow\operatorname{Ext}^1_Q(k,k)\xrightarrow{\operatorname{Ext}^1_\kappa(k,k)}\operatorname{Ext}^1_P(k,k)\\ \longrightarrow\operatorname{Ext}^1_P(Q,k)\xrightarrow{\ \delta\ }\operatorname{Ext}^2_Q(k,k)\xrightarrow{\operatorname{Ext}^2_\kappa(k,k)}\operatorname{Ext}^2_P(k,k)\end{gathered}\tag{5.2.1}$$

Next we prove $\operatorname{Im}(\delta)\subseteq\pi^2(Q)$. Indeed, $\operatorname{Tor}^P_2(k,k)$ contains no divided powers, so $\pi^2(P)$ is the subspace of k-linear functions vanishing on $\operatorname{Tor}^Q_1(k,k)^2$. Dualizing the exact sequence above, one obtains an exact sequence

$$\begin{gathered}\longrightarrow\operatorname{Tor}^P_2(k,k)\xrightarrow{\operatorname{Tor}^\kappa_2(k,k)}\operatorname{Tor}^Q_2(k,k)\xrightarrow{\operatorname{Hom}_k(\delta,k)}\operatorname{Tor}^P_1(Q,k)\\ \longrightarrow\operatorname{Tor}^P_1(k,k)\xrightarrow{\operatorname{Tor}^\kappa_1(k,k)}\operatorname{Tor}^Q_1(k,k)\longrightarrow 0\end{gathered}$$

of k-vector spaces. Since $\operatorname{Tor}^{\kappa}_*(k,k)$ is a homomorphism of algebras, it gives

$$\operatorname{Tor}^Q_1(k,k)^2 = (\operatorname{Im}(\operatorname{Tor}^{\kappa}_1(k,k))^2 \subseteq \operatorname{Im}(\operatorname{Tor}^{\kappa}_2(k,k)) = \operatorname{Ker}(\operatorname{Hom}_k(\delta,k))\,.$$

Thus, for each $\epsilon \in \operatorname{Ext}^1_P(Q,k)$ one gets $\delta(\epsilon)(\operatorname{Tor}^Q_1(k,k)^2) = 0$, as desired.

The exact sequence in the hypothesis of the lemma induces an isomorphism

$$\eth\colon \operatorname{Hom}_P(V,k) \xrightarrow{\cong} \operatorname{Ext}^1_P(Q,k)\,. \tag{5.2.2}$$

of k-vector spaces. Setting $\widetilde{\iota} = \delta\eth$, and noting that one has $\pi^1(P) = \operatorname{Ext}^1_P(k,k)$ and $\pi^1(Q) = \operatorname{Ext}^1_Q(k,k)$, one gets the desired exact sequence from that for Ext's. □

The following definition uses [2, 4.6]; see 6.6 for the standard definition.

5.3. A surjective homomorphism $\kappa\colon P \to Q$ is said to be *Golod* if the induced map $\pi^*(\kappa)\colon \pi^*(Q) \to \pi^*(P)$ is surjective and its kernel is a free Lie algebra.

When κ is Golod $\operatorname{Ker}(\pi^*(\kappa))$ is the free Lie algebra on a graded k-vector space W, with $W^i = 0$ for $i \le 1$ and $\operatorname{rank}_k W^i = \operatorname{rank}_k \operatorname{Ext}^{i-2}_P(Q,k)$ for all $i \ge 2$.

Proposition 5.4. *Let $\widetilde{V}$ denote the graded vector space with $\widetilde{V}^i = 0$ for $i \ne 2$ and $\widetilde{V}^2 = \operatorname{Hom}_P(V,k)$, let $\mathsf{T}(\widetilde{V})$ be the tensor algebra of $\widetilde{V}$, and let*

$$\iota^*\colon \mathsf{T}(\widetilde{V}) \to \operatorname{Ext}^*_Q(k,k)$$

be the unique homomorphism of graded k-algebras with $\iota^2 = \widetilde{\iota}$; see Lemma 5.2.

If β, γ, κ, and κ' are surjective homomorphisms of rings, the diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\iota"] \arrow[d, "\alpha"'] & P \arrow[r, "\kappa"] \arrow[d, "\beta"'] & Q \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & V' \arrow[r, "\iota'"] & P' \arrow[r, "\kappa'"] & Q' \arrow[r] & 0
\end{tikzcd}$$

commutes, and its rows are exact, then the following maps are equal:

$$\iota^* \circ \mathsf{T}(\operatorname{Hom}_\beta(\alpha,k)) = \operatorname{Ext}^*_\gamma(k,k) \circ \widetilde{\iota}'^*\colon \mathsf{T}(\widetilde{V}') \to \operatorname{Ext}^*_Q(k,k)\,.$$

If V is cyclic and $\iota(V)$ is contained in $\mathfrak{p}^2$, or if the homomorphism κ is Golod, then ι^* is injective, and $\operatorname{Ext}^*_Q(k,k)$ is free as a left and as a right $\mathsf{T}(\widetilde{V})$-module.

Proof. The maps $\widetilde{\iota}$ and $\widetilde{\iota}'$ are the compositions of the rows in the following diagram, which commutes by the naturality of the maps $\eth$ from (5.2.2) and δ from (5.2.1):

$$\begin{tikzcd}
\operatorname{Hom}_P(V,k) \arrow[r, "\eth"] & \operatorname{Ext}^1_P(Q,k) \arrow[r, "\delta"] & \operatorname{Ext}^2_Q(k,k) \\
\operatorname{Hom}_{P'}(V',k) \arrow[r, "\eth'"] \arrow[u, "\operatorname{Hom}_\beta(\alpha,k)"] & \operatorname{Ext}^1_{P'}(Q',k) \arrow[r, "\delta'"] \arrow[u, "\operatorname{Ext}^1_\pi(\beta,k)"'] & \operatorname{Ext}^2_{Q'}(k,k) \arrow[u, "\operatorname{Ext}^2_\gamma(k,k)"']
\end{tikzcd}$$

Set $W^2 = \iota^2(\widetilde{V}^2)$. The subalgebra $E = \iota^*(\mathsf{T}(\widetilde{V}))$ of $\operatorname{Ext}^*_Q(k,k)$ is generated by W^2. Lemma 5.2 shows that W^2 is contained in $\pi^2(Q)$, so E is the universal enveloping algebra of the Lie subalgebra ω^* of $\pi^*(Q)$, generated by W^2.

The Poincaré-Birkhoff-Witt Theorem (e.g., [3, 10.1.3.4]) implies that the universal enveloping algebra U of $\pi^*(Q)$ is free as a left and as a right E-module. Recall, from 5.1, that U equals $\operatorname{Ext}^*_Q(k,k)$. Thus, it suffices to show that ι^* is injective. This is equivalent to injectivity of ι^2 plus freeness of the associative k-algebra E; the latter condition can be replaced by freeness of the Lie algebra ω^*.

If V is contained in $\mathfrak{p}^2$, then $\operatorname{Ext}^1_\kappa(k,k)$ is surjective, so ι^2 is injective by Lemma 5.2. If V is, in addition, cyclic, then W^2 is a k-subspace of $\pi^*(Q)$, generated by a non-zero element of even degree. Any such subspace is a free Lie subalgebra.

When κ is Golod, $\pi^1(\kappa)$ is surjective by 5.3, so ι^2 is injective by Lemma 5.2. Now $\operatorname{Ker}\pi^*(\kappa)$ is a free Lie algebra, again by 5.3, hence so is its subalgebra ω^*. □

6. Cohomology of fiber products

The cohomology algebra of fiber products is known, and its structure is used in the next section. To describe it, we recall a construction of coproduct of algebras.

6.1. Let B and C be graded k-algebras, with $B^0 = k = C^0$ and $B^n = 0 = C^n$ for all $n < 0$. Thus, there exist isomorphisms $B \cong \mathsf{T}(X)/K$ and $C \cong \mathsf{T}(Y)/L$, where X and Y are graded k-vector spaces, and K and L are ideals in the respective tensor algebras, satisfying $K \subseteq X \otimes_k X$ and $L \subseteq Y \otimes_k Y$. The algebra $B \sqcup C = \mathsf{T}(X \oplus Y)/(K, L)$ is a *coproduct* of B and C in the category of graded k-algebras.

Before proceeding we fix some notation.

6.2. When $(R, \mathfrak{r}, k)$ and $(S, \mathfrak{s}, k)$ are local rings, we let $\varepsilon_R\colon R \to k$ and $\varepsilon_S\colon S \to k$ denote the canonical surjections, and form the commutative diagram

$$\xi = \begin{tikzcd} & R \arrow[dr, "\varepsilon_R"] & \\ R \times_k S \arrow[ur, "\rho"] \arrow[dr, "\sigma"'] & & k \\ & S \arrow[ur, "\varepsilon_S"'] & \end{tikzcd} \tag{6.2.1}$$

of local rings. The induced commutative diagram of graded k-algebras

$$\begin{tikzcd} & \operatorname{Ext}^*_R(k,k) \arrow[dr, "\operatorname{Ext}^*_\rho(k,k)"] & \\ k \arrow[ur] \arrow[dr] & & \operatorname{Ext}^*_{R\times_k S}(k,k) \\ & \operatorname{Ext}^*_S(k,k) \arrow[ur, "\operatorname{Ext}^*_\sigma(k,k)"'] & \end{tikzcd} \tag{6.2.2}$$

see (5.1), determines a homomorphism of graded k-algebras

$$\xi^*\colon \operatorname{Ext}^*_R(k,k) \sqcup \operatorname{Ext}^*_S(k,k) \longrightarrow \operatorname{Ext}^*_{R\times_k S}(k,k)\,. \tag{6.2.3}$$

The following result is [16, 3.4]; for k-algebras, see also [18, Ch. 3, 1.1].

6.3. The map ξ^* in (6.2.3) is an isomorphism of graded k-algebras.

To describe some invariants of modules over fiber products, we recall that the *Poincaré series* of a finite module M over a local ring $(Q, \mathfrak{q}, k)$ is defined by

$$P^Q_M = \sum_i \operatorname{rank}_k \operatorname{Ext}^i_Q(M,k)\, z^i \in \mathbb{Z}[\![z]\!]\,.$$

6.4. Dress and Krämer [7, Thm. 1] proved that each finite R-module M satisfies

$$P^{R\times_k S}_M = P^R_M \cdot \frac{P^S_k}{P^R_k + P^S_k - P^R_k P^S_k}\,.$$

Formulas for Poincaré series of S-modules are obtained by interchanging R and S.

Proposition 6.5. *Let $(R,\mathfrak{r},k)$ and $(S,\mathfrak{s},k)$ be local rings and let $\varphi\colon R\to R'$ and $\psi\colon S\to S'$ be surjective homomorphisms of rings.*

For the induced map $\varphi\times_k\psi\colon R\times_k S\to R'\times_k S'$ one has an equality

$$P^{R\times_k S}_{R'\times_k S'}=\frac{P^R_{R'}P^S_k+P^S_{S'}P^R_k-P^R_kP^S_k}{P^R_k+P^S_k-P^R_kP^S_k}.$$

Proof. Set $I=\operatorname{Ker}(\varphi)$ and $J=\operatorname{Ker}(\psi)$. The first equality below holds because one has $\operatorname{Ker}(\varphi\times_k\psi)=I\oplus J$ as ideals; the second one comes from 6.4:

$$\begin{aligned}
P^{R\times_k S}_{R'\times_k S'}&=1+z\cdot(P^{R\times_k S}_I+P^{R\times_k S}_J)\\
&=1+z\cdot\left(\frac{P^R_IP^S_k}{P^R_k+P^S_k-P^R_kP^S_k}+\frac{P^S_JP^R_k}{P^R_k+P^S_k-P^R_kP^S_k}\right)\\
&=1+\frac{z}{P^R_k+P^S_k-P^R_kP^S_k}\cdot\left(\frac{P^R_{R'}-1}{z}\cdot P^S_k+\frac{P^S_{S'}-1}{z}\cdot P^R_k\right)\\
&=\frac{P^R_{R'}P^S_k+P^S_{S'}P^R_k-P^R_kP^S_k}{P^R_k+P^S_k-P^R_kP^S_k}.
\end{aligned}$$ □

We recall Levin's [13] original definition of Golod homomorphism in terms of Poincaré series. The symbol $\preccurlyeq$ stands for termwise inequality of power series.

6.6. Every surjective ring homomorphism $R\to R'$ with $(R,\mathfrak{r},k)$ local satisfies

$$P^{R'}_k\preccurlyeq\frac{P^R_k}{1+z-zP^R_{R'}}\,,$$

see, for instance, [3, 3.3.2]. Equality holds if and only if $R\to R'$ is *Golod*.

The following result is due to Lescot [12, 4.1].

Corollary 6.7. *If φ and ψ are Golod, then so is $\varphi\times_k\psi$.*

Proof. When the homomorphisms φ and ψ are Golod the following equalities hold:

$$\begin{aligned}
\frac{1}{P^{R'\times_k S'}_k}&=\frac{1}{P^{R'}_k}+\frac{1}{P^{S'}_k}-1\\
&=\frac{(1+z-zP^R_{R'})}{P^R_k}+\frac{(1+z-zP^S_{S'})}{P^S_k}-1\\
&=\frac{(1+z-zP^R_{R'})P^S_k+(1+z-zP^S_{S'})P^R_k-P^R_kP^S_k}{P^R_kP^S_k}\\
&=\frac{(1+z)(P^R_k+P^S_k-P^R_kP^S_k)-z(P^R_kP^S_{S'}+P^S_kP^R_{R'}-P^R_kP^S_k)}{P^R_kP^S_k}\\
&=\frac{(1+z-zP^{R\times_k S}_{R'\times_k S'})\big(P^R_k+P^S_k-P^R_kP^S_k\big)}{P^R_kP^S_k}\\
&=\frac{1+z-zP^{R\times_k S}_{R'\times_k S'}}{P^{R\times_k S}_k}\,.
\end{aligned}$$

The first and last come from 6.4, the second from the definition, the penultimate one from the proposition. Stringing them together, we see that $\varphi\times_k\psi$ is Golod. □

7. Cohomology of connected sums

We compute the cohomology algebra of a connected sum of local rings over certain Golod homomorphisms, using amalgams of graded k-algebras.

7.1. Let $\beta\colon B \leftarrow A \to C : \gamma$ be homomorphisms of graded k-algebras.

Let $B \sqcup_A C$ denote the quotient of the coproduct $B \sqcup C$, see 6.1, by the two-sided ideal generated by the set $\{\beta(a) - \gamma(a) \mid a \in A\}$. It comes equipped with canonical homomorphisms of graded k-algebras $\gamma'\colon B \to B \sqcup_A C \leftarrow C : \beta'$, satisfying $\gamma'\beta = \beta'\gamma$. The universal property of coproducts implies that $B \sqcup_A C$ is an *amalgam* of β and γ in the category of graded k-algebras.

If B and C are free as left graded A-modules and as right graded A-modules, then Lemaire [11, 5.1.5 and 5.1.10] shows that the maps γ' and β' are injective, and

$$\frac{1}{H_{B \sqcup_A C}} = \frac{1}{H_B} + \frac{1}{H_C} - \frac{1}{H_A}\,. \tag{7.1.1}$$

7.2. Given a connected sum diagram (2.0.1) with local rings $(R, \mathfrak{r}, k)$ and $(S, \mathfrak{s}, k)$, $T = k$, and canonical surjection ε_R and ε_S, set $R' = R/\iota_R(V)$ and $S' = S/\iota_S(V)$.

We refine (2.0.1) to a commutative diagram

$$\Xi = \begin{tikzcd}
 & & R \arrow[rr, two heads, "\varphi"'] \arrow[rrrr, bend left, two heads, "\varepsilon_R"] & & R' \arrow[drr, two heads] & & \\
V \arrow[urr, bend left, "\iota_R"] \arrow[r, "\iota"] \arrow[drr, bend right, "\iota_S"'] & R \times_k S \arrow[ur, two heads, "\rho"] \arrow[rr, two heads, "\kappa"] \arrow[dr, two heads, "\sigma"'] & & R\#_k S \arrow[ur, two heads, "\rho'"] \arrow[rr, two heads, "\varkappa"] \arrow[dr, two heads, "\sigma'"'] & & R' \times_k S' \arrow[ul, two heads] \arrow[dl, two heads] \arrow[r, two heads] & k \\
 & & S \arrow[rr, two heads, "\psi"] \arrow[rrrr, bend right, two heads, "\varepsilon_S"'] & & S' \arrow[urr, two heads] & &
\end{tikzcd} \tag{7.2.1}$$

where $\iota(v) = (\iota_R(v), \iota_S(v))$ and two-headed arrows denote canonical surjections.

Proposition 5.4 now gives a commutative diagram of graded k-algebras:

$$\begin{tikzcd}
 & & \mathrm{Ext}^*_{R'}(k,k) \arrow[dl, "\mathrm{Ext}^*_{\rho'}(k,k)"'] \arrow[dr] & \\
\mathsf{T}(\widetilde{V}) \arrow[urr, bend left, "\iota_R^*"] \arrow[r, "\iota^*"] \arrow[drr, bend right, "\iota_S^*"'] & \mathrm{Ext}^*_{R\#_k S}(k,k) & & \mathrm{Ext}^*_{R' \times_k S'}(k,k) \arrow[ll, "\mathrm{Ext}^*_{\varkappa}(k,k)"'] \\
 & & \mathrm{Ext}^*_{S'}(k,k) \arrow[ul, "\mathrm{Ext}^*_{\sigma'}(k,k)"] \arrow[ur] &
\end{tikzcd} \tag{7.2.2}$$

By 7.1, the preceding diagram defines a homomorphism of graded k-algebras

$$\Xi^*\colon\ \mathrm{Ext}^*_{R'}(k,k) \sqcup_{\mathsf{T}(\widetilde{V})} \mathrm{Ext}^*_{S'}(k,k) \longrightarrow \mathrm{Ext}^*_{R\#_k S}(k,k)\,. \tag{7.2.3}$$

Theorem 7.3. *Assume that ι_R and ι_S in 7.2 are injective and non-zero.*

If the homomorphism $\varkappa\colon R\#_k S \to R' \times_k S'$ is Golod, in particular, if

(a) *the rings R and S are Gorenstein of length at least 3, or*
(b) *the homomorphisms φ and ψ are Golod,*

then Ξ^* *in* (7.2.3) *is an isomorphism, and the canonical maps below are injective:*

$$\operatorname{Ext}^*_{R'}(k,k) \xrightarrow{\operatorname{Ext}^*_{\rho'}(k,k)} \operatorname{Ext}^*_{R\#_k S}(k,k) \xleftarrow{\operatorname{Ext}^*_{\sigma'}(k,k)} \operatorname{Ext}^*_{S'}(k,k)\,.$$

Corollary 7.4. *When* $\varkappa$ *is Golod, for every* R'*-module* N *one has*

$$P^{R\#_k S}_N = P^{R'}_N \cdot \frac{P^{S'}_k}{P^{R'}_k + P^{S'}_k - (1-rz^2)\cdot P^{R'}_k P^{S'}_k}\,,$$

where $r = \operatorname{rank}_k V$ *(and thus,* $r=1$ *under condition* (a)*). Formulas for Poincaré series of* S'*-modules are obtained by interchanging* R' *and* S'.

In preparation for the proofs, we review a few items.

7.5. When $(P,\mathfrak{p},k)$ is a local ring and $\kappa\colon P\to Q$ a surjective homomorphism with $\mathfrak{p}\operatorname{Ker}(\kappa)=0$, the following inequality holds, with equality if and only if κ is Golod:

$$P^Q_k \preccurlyeq \frac{P^P_k}{1-\operatorname{rank}_k(\operatorname{Ker}(\kappa))\cdot z^2\cdot P^P_k}\,.$$

Indeed, the short exact sequence of P-modules $0\to\operatorname{Ker}(\kappa)\to P\to Q\to 0$ yields $P^P_Q = 1+\operatorname{rank}_k(\operatorname{Ker}(\kappa))\cdot z\cdot P^P_k$, so the assertion follows from 6.6.

The Golod property may be lost under composition or decomposition, but:

Lemma 7.6. *Let* $P \xrightarrow{\kappa} Q \xrightarrow{\varkappa} P'$ *be surjective homomorphisms of rings.*
When $\mathfrak{p}\operatorname{Ker}(\varkappa\kappa)=0$ *holds, the map* $\varkappa\kappa$ *is Golod if and only if* $\varkappa$ *and* κ *are.*

Proof. Set $\operatorname{rank}_k\operatorname{Ker}(\kappa)=r$ and $\operatorname{rank}_k\operatorname{Ker}(\varkappa)=r'$. From 7.5 one gets

$$P^{P'}_k \preccurlyeq \frac{P^Q_k}{1-r'z^2\cdot P^Q_k} \preccurlyeq \frac{\dfrac{P^P_k}{1-rz^2\cdot P^P_k}}{1-r'z^2\cdot\dfrac{P^P_k}{1-rz^2\cdot P^P_k}} = \frac{P^P_k}{1-(r+r')z^2\cdot P^P_k}\,.$$

One has $\operatorname{rank}_k\operatorname{Ker}(\varkappa\kappa)=r+r'$, so the desired assertion follows from 7.5. □

7.7. When $(Q,\mathfrak{q},k)$ is an artinian Gorenstein ring with $\operatorname{edim}Q\geq 2$, the canonical surjection $Q\to Q/\operatorname{Soc}Q$ is a Golod homomorphism; see [15, Theorem 2].

Proof of Theorem 7.3. For $Q=R\#_k S$ and $P'=R'\times_k S'$, we have a commutative diagram, with θ the canonical surjection, see 7.1, and ξ^* the bijection from 6.3:

$$\begin{tikzcd}
\operatorname{Ext}^*_{R'}(k,k)\sqcup_{\mathsf{T}(\widetilde V)}\operatorname{Ext}^*_{S'}(k,k) \arrow[r, "\Xi^*"] & \operatorname{Ext}^*_Q(k,k) \\
\operatorname{Ext}^*_{R'}(k,k)\sqcup\operatorname{Ext}^*_{S'}(k,k) \arrow[u, "\theta"] \arrow[r, "\cong", "\xi^*"'] & \operatorname{Ext}^*_{P'}(k,k) \arrow[u, "\operatorname{Ext}^*_{\varkappa}(k,k)"']
\end{tikzcd}$$

The map $\operatorname{Ext}^*_{\varkappa}(k,k)$ is surjective because $\varkappa$ is Golod, see 5.3, so Ξ^* is surjective.

Set $D=\operatorname{Ext}^*_{R'}(k,k)\sqcup_{\mathsf{T}(\widetilde V)}\operatorname{Ext}^*_{R'}(k,k)$. By Proposition 5.4, ι^*_R, ι^*, and ι^*_S turn their targets into free graded $\mathsf{T}(\widetilde V)$-modules, left and right, so (7.1.1) gives:

$$\frac{1}{H_D} = \frac{1}{P^{R'}_k}+\frac{1}{P^{S'}_k}-\frac{1}{H_{\mathsf{T}(\widetilde V)}} = \frac{1}{P^{R'}_k}+\frac{1}{P^{S'}_k}-(1-rz^2)\,.$$

On the other hand, from 7.5 and 6.4 we obtain

$$\frac{1}{P_k^Q} = \frac{1}{P_k^{P'}} + rz^2 = \left(\frac{1}{P_k^{R'}} + \frac{1}{P_k^{S'}} - 1\right) + rz^2\,. \tag{7.8.1}$$

Thus, one has $H_D = P_k^Q$. This implies that the surjection Ξ^* is an isomorphism.

The injectivity of $\operatorname{Ext}^*_{\rho'}(k,k)$ and $\operatorname{Ext}^*_{\sigma'}(k,k)$ now results from Proposition 5.4.

It remains to show that condition (a) or (b) implies that $\varkappa$ is Golod.

(a) Let R and S be artinian Gorenstein of length at least 3. The socle of R is equal to the maximal non-zero power of $\mathfrak{r}$, and $\mathfrak{r}^2 = 0$ would imply $\ell(R) = 2$, so we have $\operatorname{Soc} R \subseteq \mathfrak{r}^2$. By symmetry, we also have $\operatorname{Soc} S \subseteq \mathfrak{s}^2$.

Set $P = R \times_k S$. By definition, Q equals P/pP, where p is a non-zero element in $\operatorname{Soc} P$. The maximal ideal $\mathfrak{p}$ of P is equal to $\mathfrak{r} \oplus \mathfrak{s}$, so $\operatorname{Soc} P = \operatorname{Soc} R \oplus \operatorname{Soc} S$ is in $\mathfrak{r}^2 \oplus \mathfrak{s}^2$. This gives the equality below, and (1.5.1) the first inequality:

$$\operatorname{edim} Q = \operatorname{edim} P \geq \operatorname{edim} R + \operatorname{edim} S - \operatorname{edim} k \geq 2\,.$$

Since the ring Q is artinian Gorenstein by Theorem 2.8, and the kernel of the map $Q \to P'$ is non-zero and is in $\operatorname{Soc} Q$, this homomorphism is a Golod by 7.7.

(b) If φ and ψ are Golod, then so is $\varphi \times_k \psi$ by Corollary 6.7. From the equality $\varphi \times_k \psi = \varkappa\kappa$ and Lemma 7.6, one concludes that $\varkappa$ is Golod. □

Proof of Corollary 7.4. As $\operatorname{Ext}_{\rho'}(k,k)$ is injective, the first equality in the string

$$P_N^Q = P_N^{R'} \cdot \frac{P_k^Q}{P_k^{R'}} = P_N^{R'} \cdot \frac{P_k^{S'}}{P_k^{R'} + P_k^{S'} - (1 - rz^2)P_k^{R'}P_k^{S'}}$$

follows from a result of Levin; see [14, 1.1]. The second one comes from (7.8.1). □

8. Indecomposable Gorenstein rings

In this section we approach the problem of identifying Gorenstein rings that cannot be decomposed in a non-trivial way as a connected sum of Gorenstein local rings. Specifically, we prove that complete intersection rings have no such decomposition over regular rings, except in a single, well understood special case.

Recall that, by Cohen's Structure Theorem, the $\mathfrak{r}$-adic completion $\widehat{R}$ of a local ring$(R,\mathfrak{r},k)$ is isomorphic to $\widetilde{R}/K$, with $(\widetilde{R},\widetilde{\mathfrak{r}},k)$ regular local and $K \subseteq \widetilde{\mathfrak{r}}^2$. One says that R is *complete intersection* (*of codimension* c) if K can be generated by a $\widetilde{R}$-regular sequence (of length c). A *hypersurface* ring is a complete intersection ring of codimension 1; it is *quadratic* in case K is generated by an element in $\widetilde{\mathfrak{r}}^2 \smallsetminus \widetilde{\mathfrak{r}}^3$.

We also need homological characterizations of complete intersection rings:

8.1. A local ring $(R,\mathfrak{r},k)$ is complete intersection if and only if $\pi^3(R) = 0$, if and only if $P_k^R(z) = (1+z)^b/(1-z)^c$ with $b, c \in \mathbb{Z}$, see [9, 3.5.1].

If R is complete intersection, then $\operatorname{codim} R = \operatorname{rank}_k \pi^2(R) = c$; see [9, 3.4.3].

Now we return to the setup and notation of Section 2, which we recall:

8.2. The rings in the diagram (2.0.1) are local: $(R,\mathfrak{r},k)$, $(S,\mathfrak{s},k)$ and $(T,\mathfrak{t},k)$.

The maps ε_R and ε_S are surjective; set $I = \operatorname{Ker}(\varepsilon_R)$ and $J = \operatorname{Ker}(\varepsilon_S)$.

The maps ι_R and ι_S are injective.

Theorem 8.3. *When R and S are Gorenstein of dimension d, T is regular of dimension d, and $\iota_R(V) = (0 : I)$ and $\iota_S(V) = (0 : J)$, the ring $R\#_T S$ is a local complete intersection if and only if one of the following conditions holds:*

(a) *R is a quadratic hypersurface ring and S is a complete intersection ring. In this case, $R\#_T S \cong S$.*
(b) *S is a quadratic hypersurface ring and R is a complete intersection ring. In this case, $R\#_T S \cong R$.*
(c) *R and S are non-quadratic hypersurface rings. In this case, $\operatorname{codim}(R\#_T S) = 2$.*

Proof. Let $(P, \mathfrak{p}, k)$ denote the local ring $R \times_T S$, see Lemma 1.2, and $Q = R\#_T S$.

If $e(R) = 1$ or $e(S) = 1$, then $R\#_T S = 0$ by Proposition 2.6 and Theorem 2.8. Else, the ring Q is local, see 2.2; let $\mathfrak{q}$ denote its maximal ideal.

If $e(R) = 2$, then $Q \cong S$ by Proposition 2.6, so Q and S are complete intersection simultaneously. The case $e(S) = 2$ is similar, so we assume $e(R) \geq 3$ and $e(S) \geq 3$.

The P-modules P, Q, R, S, and T are Cohen-Macaulay of dimension d; see Proposition 1.7 and Theorem 2.8. Tensoring the diagram (2.0.1) with $P[y]_{\mathfrak{p}[y]}$ over P, we may assume that k is infinite. Choose a sequence $\boldsymbol{x}$ in P that is regular on P and T and satisfies $\ell(T/\boldsymbol{x}T) = e(T)$; see 2.5. Since T is a regular ring, we have $e(T) = 1$, hence $T/\boldsymbol{x}T = k$, so the image of $\boldsymbol{x}$ in T is a minimal set of generators of $\mathfrak{t}$. The surjective homomorphism $Q \to T$ induces a surjective k-linear map $\mathfrak{q}/\mathfrak{q}^2 \to \mathfrak{t}/\mathfrak{t}^2$, so the image of $\boldsymbol{x}$ in Q extends to a minimal generating set of $\mathfrak{q}$.

Since $\boldsymbol{x}$ is a system of parameters for P, and Q, R, and S are d-dimensional Cohen-Macaulay P-modules, $\boldsymbol{x}$ is also a system of parameters for each one of them. Thus, $\boldsymbol{x}$ is a regular sequence on Q, R, and S. Since $\boldsymbol{x}$ is part of a minimal set of generators of $\mathfrak{q}$, the ring Q is complete intersection of codimension c if and only if so is $Q/\boldsymbol{x}Q$. Also, R and S are Gorenstein if and only so are $R/\boldsymbol{x}R$ and $S/\boldsymbol{x}S$, and they satisfy $\ell(R) \geq e(R) \geq 3$ and $\ell(S) \geq e(S) \geq 3$; see 2.5. Lemma 2.3 gives an isomorphism of rings $Q/\boldsymbol{x}Q \cong (R/\boldsymbol{x}R)\#_k(S/\boldsymbol{x}S)$. Thus, after changing notation, for the rest of the proof we may assume $Q = R\#_k S$, where R and S are artinian Gorenstein rings that are not quadratic hypersurfaces.

Let Q be complete intersection and assume $\operatorname{edim} R \geq 2$. Set $R' = R/\operatorname{Soc} R$. Theorem 7.3 shows that the homomorphism $Q \to R'$ induces an injective homomorphism of cohomology algebras, and hence one of homotopy Lie algebras; see 5.1. This gives the second inequality in the following string, where the first inequality comes from 5.3 (because $R \to R'$ is Golod by 7.7), and the equality from 8.1:

$$\operatorname{rank}_k \operatorname{Ext}^1_R(R', k) \leq \operatorname{rank}_k \pi^3(R') \leq \operatorname{rank}_k \pi^3(Q) = 0\,.$$

It follows that R' is free as an R-module. On the other hand, it is annihilated by $\operatorname{Soc} R$, and this ideal is non-zero because R is artinian. This contradiction implies $\operatorname{edim} R = 1$, so R is a hypersurface ring. By symmetry, so is S.

Conversely, if R and S are hypersurface rings, then Corollary 7.4 gives

$$P^Q_k = \frac{1}{1-z} \cdot \frac{\dfrac{1}{1-z}}{\dfrac{1}{1-z} + \dfrac{1}{1-z} - (1-z^2)\cdot\dfrac{1}{1-z}\cdot\dfrac{1}{1-z}} = \frac{1}{(1-z)^2}\,.$$

This implies that Q is a complete intersection ring of codimension 2; see 8.1. $\square$

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