ON THE DISCRIMINANT OF TWISTED TENSOR PRODUCTS

JASON GADDIS, ELLEN KIRKMAN, W. FRANK MOORE

Abstract. We provide formulas for computing the discriminant of noncommutative algebras over central subalgebras in the case of Ore extensions and skew group extensions. The formulas follow from a more general result regarding the discriminants of certain twisted tensor products. We employ our formulas to compute automorphism groups for examples in each case.

1. Introduction

Throughout $\Bbbk$ is an algebraically closed, characteristic zero field and all algebras are $\Bbbk$-algebras. All unadorned tensor products should be regarded as over $\Bbbk$. Given an algebra R, we denote by $R^\times$ the set of units in R. If $\sigma \in \operatorname{Aut}(R)$, then R^σ denotes the subalgebra of elements of R that are fixed under σ. We denote the center of R by $C(R)$.

Automorphism groups of commutative and noncommutative algebras can be notoriously difficult to compute. For example, $\operatorname{Aut}(\Bbbk[x,y,z])$ is not yet fully understood. In [2], the authors give a method for determining the automorphism groups of noncommutative algebras using the discriminant. This was studied further in [3, 4, 5, 6]. Discriminants of deformations of polynomial rings were computed using Poisson geometry in [11, 13].

We refer the reader to [2] for the general definitions of trace and discriminant in the context of noncommutative algebras. We review the definitions only in the case that B is an algebra finitely generated free over a central subalgebra $R \subseteq C(B)$ of rank n.

Left multiplication defines a natural embedding $\operatorname{lm} : B \to \operatorname{End}_C(B) \cong M_n(R)$. The usual matrix trace defines a map $\operatorname{tr}_{\text{int}} : M_n(R) \to R$ called the internal trace. The regular trace is defined as the composition $\operatorname{tr}_{\text{reg}} : B \xrightarrow{\operatorname{lm}} M_n(R) \xrightarrow{\operatorname{tr}_{\text{int}}} R$. For our purposes, tr will be $\operatorname{tr}_{\text{reg}}$.

Let ω be a fixed integer and $Z := \{z_i\}_{i=1}^{\omega}$ a subset of B. The discriminant of Z is defined to be

$$d_\omega(Z) = \det(\operatorname{tr}(z_i z_j))_{\omega\times\omega} \in R.$$

If Z is an R-basis of B, then the discriminant of B over R is defined to be

$$d(B/R) =_{R^\times} d_\omega(Z),$$

where $x =_{R^\times} y$ means $x = cy$ for some $c \in R^\times$.

The discriminant is independent of R-linear bases of B [2, Proposition 1.4]. Moreover, if $\phi \in \operatorname{Aut}(B)$ and ϕ preserves R, then ϕ preserves the ideal generated by $d(B/R)$ [2, Lemma 1.8].

Computing the discriminant is a computationally difficult task, even for algebras with few generators. For example, the matrix obtained from $\operatorname{tr}(z_i z_j)$ for the skew group algebra $\Bbbk_{-1}[x_1, x_2, x_3]\#\mathcal{S}_3$ has size 288×288. Our first goal is to provide methods for obtaining the discriminant in cases where the algebra may be realized as an extension of a smaller algebra where computations may be easier.

If A is an algebra and $\sigma \in \operatorname{Aut}(A)$, then the Ore extension $A[t;\sigma]$ is generated by A and t with the rule $ta = \sigma(a)t$ for all $a \in A$.

Theorem 1 (Theorem 6.1)**.** *Let A be an algebra and set $S = A[t;\sigma]$, where $\sigma \in \operatorname{Aut}(A)$ has order $m < \infty$ and no σ^i, $1 \le i < m$, is inner. Suppose $R \subseteq C(S)$ and set $B = R \cap A^\sigma$. If A is free over B of rank n, then*

$$d(S/R) =_{R^\times} (d(A/B))^m \left(t^{m-1}\right)^{mn}.$$

We say an automorphism σ of A is inner if there exists $a \in A$ such that $xa = a\sigma(x)$ for all $x \in A$. This is not the standard definition of an inner automorphism but it agrees if a is a unit because then $a^{-1}ba = \sigma(b)$. Since a is normal and assuming A is a domain, one can localize at the Ore set of powers of a so as to get an inner automorphism in $A[a^{-1}]$. When σ^k is an inner automorphism for some $1 \le k < m = |\sigma|$ then the center of $A[t;\sigma]$ can be larger than $(C(A) \cap A^\sigma)[t^m]$. We denote the set of all inner automorphisms of A by $\operatorname{Inn}(A)$. It is a routine verification that $\operatorname{Inn}(A)$ forms a subgroup of $\operatorname{Aut}(A)$.

Let A be an algebra and G a finite group that acts on A as automorphisms. Denote by $\Bbbk G$ the group algebra of G. The skew group algebra $A\#G$ has the underlying set $A \otimes \Bbbk G$ and multiplication defined by

$$(a \otimes g)(b \otimes h) = a(g.b) \otimes gh \text{ for all } a, b \in A, \quad g, h \in G.$$

The natural embedding

$$A \to A\#G$$
$$a \mapsto a \otimes e$$

where e is the identity of G, allows us to identify A with its image in $A\#G$. If G contains no non-identity inner automorphisms and acts faithfully on A, then by Lemma 2.2, $C(A\#G) = C(A)^G$ under the above identification.

Theorem 2 (Theorem 7.1)**.** *Let A be an algebra and G a finite group that acts on A as automorphisms such that no non-identity element of G is inner. Set $S = A\#G$ and identify A with its image under the embedding $a \mapsto a \otimes e$. Suppose A is a finitely generated free over $R \subseteq C(A)^G$. Then*

$$d(S/R) =_{R^\times} d(A/R)^{|G|}.$$

The condition that A is a finitely generated free R-module is satisfied in case A is a commutative polynomial ring and G is a group generated by reflections by the classical results of Chevalley [7] and Shephard-Todd [14]. Section 4 is devoted to showing that such discriminants may be computed in a manner similar to the discriminant of an algebraic number field; see Proposition 4.3.

Both Ore extensions (by an automorphism) and skew group algebras are examples of *twisted tensor products*. We prove a more general formula regarding discriminants of certain twisted tensor products from which the two previous theorems follow in Section 5. The necessary background for these results is in Sections 2 and 3. We then apply this result to the case of Ore extensions (Section 6) and skew group algebras (Section 7), as well as provide examples of each.

Finally, in Section 8 we use the results on discriminants to compute the automorphism groups of some Ore extensions and skew group algebras. Each section contains several examples using these formulas to compute discriminants. Many computations herein were assisted by routines in Macaulay2 using the NCAlgebra package[1].

2. Twisted tensor products and monoid algebras

Let A and B be algebras and let $\tau : B\otimes A \to A\otimes B$ be a $\Bbbk$-linear homomorphism subject to the conditions that $\tau(b \otimes 1_A) = 1_A \otimes b$ and $\tau(1_B \otimes a) = a \otimes 1_B$, $a \in A$, $b \in B$. A multiplication on $A \otimes B$ is then given by $\mu_\tau := (\mu_A \otimes \mu_B) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B)$ where μ_A and μ_B are the multiplication maps on A and B, respectively. By [1, Proposition 2.3], μ_τ is associative if and only if $\tau \circ (\mu_B \otimes \mu_A) = \mu_\tau \circ (\tau \otimes \tau) \circ (\mathrm{id}_B \otimes \tau \otimes \mathrm{id}_A)$ as maps $B \otimes B \otimes A \otimes A \to A \otimes B$. The triple $(A \otimes B, \mu_\tau)$ is a twisted tensor product of A and B, denoted by $A \otimes_\tau B$.

We are concerned with twisted tensor products when B is the monoid algebra of a monoid M that acts on A as automorphisms. Let M be a monoid and $\rho : M \longrightarrow \mathrm{Aut}(A)$ a monoid homomorphism (so that $\rho(mm') = \rho(m)\rho(m')$ and $\rho(e_M) = \mathrm{id}_A$ where e_M denotes the identity of M). For $m \in M$ and $a \in A$, we write ma for $\rho(m)(a)$. For a monoid M, we let $C(M)$ denote the center of M.

In this case, one may check that since ρ is a homomorphism, the assignment

$$\begin{aligned} \tau : k[M] \otimes A \quad &\longrightarrow \quad A \otimes k[M] \\ m \otimes a \quad &\mapsto \quad ma \otimes m \end{aligned}$$

extends linearly to a $\Bbbk$-linear map that makes the multiplication μ_τ associative. We will denote such a twisted tensor product by $A \otimes_\tau k[M]$ when the homomorphism $\rho : M \longrightarrow \mathrm{Aut}(A)$ is understood. We identify the elements of A, $k[M]$ and M with their images in $A \otimes_\tau k[M]$ under the canonical embeddings

$$A \longrightarrow A \otimes_\tau k[M] \longleftarrow k[M] \longleftarrow M.$$

For such an action of a monoid M on an algebra A, we let A^M denote the set

$$A^M = \{a \in A \mid ma = a \text{ for all } m \in M\}.$$

It is easy to check that A^M is a subalgebra of A. Furthermore, since the center of an algebra is preserved under any automorphism, M acts on $C(A)$ as well, so one may also consider $C(A)^M$.

[1] Available at http://users.wfu.edu/moorewf.

Remark 2.1. The two main applications of interest are Ore extensions (by an automorphism) and skew group algebras, and each fit into this framework. In the case of an Ore extension $A[t;\sigma]$ for an automorphism σ of A, $A[t;\sigma] \cong A \otimes_\tau \Bbbk[\mathbb{N}]$ where $\rho : \mathbb{N} \longrightarrow \mathrm{Aut}(A)$ sends 1 to σ. Similarly, a finite group G acting on A as automorphisms is the same as a group homomorphism $\rho : G \longrightarrow \mathrm{Aut}(A)$, and one may check that $A\#G = A \otimes_\tau \Bbbk[G]$.

To better explain some of the hypotheses we will need in our theorem regarding discriminants of twisted tensor products, we must discuss the center of $A \otimes_\tau k[M]$.

Lemma 2.2. *Let A be an algebra, M a monoid that acts on A through the monoid homomorphism $\rho : M \longrightarrow \mathrm{Aut}(A)$, and $H = \ker\rho$. Suppose that $H \subseteq C(M)$ and $\mathrm{im}\,\rho \cap \mathrm{Inn}(A) = \{\mathrm{id}_A\}$. Then $C(A \otimes_\tau k[M]) = C(A)^M \otimes k[H]$.*

Proof. Let $T = A \otimes_\tau k[M]$ and choose $\sum_{m\in M} a_m \otimes m \in C(T)$. Then for all $x \in A$ we have

$$\left(\sum_{m\in M} a_m \otimes m\right)(x \otimes e_M) = \sum_{m\in M} a_m(mx) \otimes m.$$

On the other hand,

$$(x \otimes e_M)\left(\sum_{m\in M} a_m \otimes m\right) = \sum_{m\in M} xa_m \otimes m.$$

Since M acts as automorphisms, each nonzero a_m is a normal element corresponding to the automorphism induced by m. This implies that if $a_m \neq 0$, then m induces an inner automorphism of A, and hence by hypothesis $m \in \ker\rho$. Note that in this case it also follows that $a_m \in C(A)$, so that each term in the sum $\sum_{m\in M} a_m \otimes m$ has $a_m \in C(A)$ and $m \in H$.

Now for all $m' \in M$, one has

$$\left(\sum_{m\in H} a_m \otimes m\right)(1 \otimes m') = \sum_{m\in M} a_m \otimes mm'.$$

On the other hand,

$$\begin{aligned}(1 \otimes m')\left(\sum_{m\in H} a_m \otimes m\right) &= \sum_{m\in M} m'a_m \otimes m'm \\ &= \sum_{m\in M} m'a_m \otimes mm'\end{aligned}$$

where the second equality follows since $H \subseteq \ker\rho$. Therefore we have that $m'a_m = a_m$ for all $m' \in M$, hence $a_m \in C(A)^M$. The claim now follows. $\square$

We note that the hypothesis $H \subseteq \ker\rho$ in Lemma 2.2 is trivially satisfied when M is a group and M acts faithfully on A (as is the case in most skew group algebra computations), as well as when $\mathbb{N}$ acts on A as an automorphism σ (as in the Ore extension case).

The hypothesis $\operatorname{im}\rho \cap \operatorname{Inn}(A) = \{\mathrm{id}_A\}$ is a bit more restrictive however. Indeed, extending Lemma 2.2 to the case where a non-identity element of M acts as an inner automorphism is nontrivial in general, but can be done for an Ore extension of a domain. For an automorphism σ of A we define

$$N(\sigma) = \{a \in A^\sigma : xa = a\sigma(x) \text{ for all } x \in A\}.$$

Note that $N(\sigma) = \{0\}$ when σ is not an inner automorphism induced by an element of A^σ, $N(1_A) = A^\sigma \cap C(A)$, and if $|\sigma| = m$ then $N(\sigma^\ell) = N(\sigma^i)$, where $\ell \equiv i \mod m$.

Lemma 2.3. *Let A be a domain, σ an automorphism of A with $|\sigma| = m$, and set $S = A[t;\sigma]$. Then $C(S) = \oplus N(\sigma^i)[t^i]$, where $N(\sigma^i)$ is nonzero if σ^i an inner automorphism induced by an element of A^σ.*

Proof. The containment $\supseteq$ is clear, so suppose $\sum a_i t^i \in C(S)$; then by the t-grading, $a_i t^i \in C(S)$ for each i. Hence,

$$\left(\sum \sigma(a_i)t^i\right) t = t\left(\sum a_i t^i\right) = \left(\sum a_i t^i\right) t.$$

Thus $\sigma(a_i) = a_i$ for each i, and $a_i \in A^\sigma$. For $a_i t^i \in C(S)$ and any $b \in A$, we have $ba_i t^i = a_i t^i b = a_i \sigma^i(b) t^i$. Because S is a domain, then $ba_i = a_i\sigma^i(b)$ for all i, and $a_i \in N(\sigma^i)$. Hence, for $i \not\equiv 0 \mod m$ we have that σ^i is a non-identity inner automorphism induced by a_i. □

Before closing this section, we will need a lemma that characterizes when an extension of a monoid algebras is a free extension. Before stating the lemma, we need a definition.

Definition 2.4. A monoid M is left cancellative provided for all $a, b, c \in M$, the equality $ab = ac$ implies that $b = c$. One similarly defines right cancellative monoid, and M is called cancellative if M is both left and right cancellative.

Lemma 2.5. *Let H be a submonoid of a cancellative monoid M, and let $\mathcal{B} = \{m_1, \dots, m_\ell\}$ be a subset of M. Then $\mathcal{B}$ is a basis of $\Bbbk[M]$ as a left (respectively right) $\Bbbk[H]$-module if and only if the right (respectively left) cosets of H represented by all the elements of $\mathcal{B}$ are disjoint, and M is the union of the right (respectively left) cosets of H represented by all the elements of $\mathcal{B}$.*

Proof. Note that $\mathcal{B}$ is a basis of $\Bbbk[M]$ as a left $\Bbbk[H]$-module if and only if for each $m \in M$ there exists a unique $h \in H$ and $1 \leq i \leq \ell$ such that $m = hm_i$. This is precisely the statement that the right cosets of H by all the elements of $\mathcal{B}$ are disjoint and cover M. □

A source of examples of cancellative monoids are submonoids of groups. We record here a lemma that we will use later regarding extensions of monoid algebras in this context.

Lemma 2.6. *Let M be a monoid which embeds in a group G, K a group, $\rho : G \longrightarrow K$ a group homomorphism, and $H = \ker\rho \cap M$. Suppose that, for all $g \in \ker\rho$, one has $g \in M$ or $g^{-1} \in M$. Then:*

(a) For all $m_1, m_2 \in M$, $\rho(m_1) = \rho(m_2)$ if and only if either $Hm_1 \subseteq Hm_2$ or $Hm_2 \subseteq Hm_1$, and similarly for left cosets.

(b) Suppose that $\Bbbk[M]$ is a free left or right $\Bbbk[H]$-module with basis $\{m_1, \ldots, m_\ell\}$ with $m_i \in M$ and $m_1 = e_M$. Then for every $j = 1, \ldots, \ell$, there exists a unique i such that $m_i m_j \in H$.

Proof. We prove each of the claims in the case of right cosets and a left module structure as in Lemma 2.5.

It is easy to see that either containment of cosets above implies $\rho(m_1) = \rho(m_2)$. Conversely, suppose that $\rho(m_1) = \rho(m_2)$. Then $m_1 m_2^{-1} \in \ker \rho$ so by hypothesis either $m_1 m_2^{-1} \in H$ or $m_2 m_1^{-1} \in H$. In the first case, one has $Hm_1 \subseteq Hm_2$ and in the second case, one has $Hm_2 \subseteq Hm_1$, proving part (a).

For the second claim, Lemma 2.5 shows that given any element $m \in M$, there exists a unique i such that $m \in Hm_i$. We will denote the assignment $m \mapsto m_i$ by the notation $\operatorname{rep}(m)$. Fix a $j = 1, \ldots, n$ and consider the following function defined on the cosets of the basis in the hypothesis:

$$\begin{aligned} \Phi_j : \{Hm_1, \ldots, Hm_\ell\} &\to \{Hm_1, \ldots, Hm_\ell\} \\ Hm_i &\mapsto H \operatorname{rep}(m_i m_j). \end{aligned}$$

Suppose that $\Phi_j(m_i) = \Phi_j(m_{i'})$. Then one has that $m_i m_j$ and $m_{i'} m_j$ are in the same coset Hm_k for some k. It follows that $\rho(m_i) = \rho(m_{i'})$ and hence by part (a) one has that Hm_i and $Hm_{i'}$ intersect nontrivially. Therefore by Lemma 2.5 one has $m_i = m_{i'}$. This shows that Φ_j is one-to-one and hence onto. Since we chose $m_1 = e_M$, one of the cosets is H hence given any j, there is a unique i such that $m_i m_j \in H$ as claimed. $\square$

Note that $\mathbb{N}$ is a submonoid of the group $\mathbb{Z}$ that intersects every subgroup of $\mathbb{Z}$ nontrivially, and hence fits into the framework of each of the two previous lemmas. Furthermore, if M is a group then each of the two lemmas' hypotheses hold trivially. Therefore, these hypotheses are not restrictive from the point of view of our intended applications.

3. Background on discriminants

In this section, we assume that A/B be a free extension of algebras with $B \subseteq C(A)$.

Notation 3.1. Suppose $\sigma \in \operatorname{Aut}(A)$ and that σ restricts to the identity on B. Then $\sigma : A \longrightarrow A$ is a B-module homomorphism. Therefore, given a basis $\{x_1, \ldots, x_g\}$ of A as a B-module, we can represent σ using a matrix with entries in B which we denote X_σ. That is, if $x \in A$ and the coordinate vector of x with respect to the chosen basis is $\boldsymbol{x}$, then $\sigma(x) = X_\sigma \boldsymbol{x}$. Note that the determinant of X_σ is independent of the chosen basis of A over B, and is an element of $B^\times$. We therefore denote $\det X_\sigma$ by $\det_{A/B} \sigma$.

If $\rho : M \longrightarrow \operatorname{Aut}(A)$ is a monoid homomorphism and $m \in M$, we write X_m and $\det_{A/B} m$ to denote $X_{\rho(m)}$ and $\det_{A/B} \rho(m)$, respectively.

In the following definition we consider $\sigma \in \mathrm{Aut}(A)$ that restricts to the identity on B to twist the standard trace pairing. This will be necessary for our calculations that appear in Section 5. Recall that for an algebra A, we use μ_A to denote multiplication on A.

Definition 3.2. Let $\sigma \in \mathrm{Aut}(A)$ such that σ restricts to the identity on B. Define the trace form of the extension A/B twisted by σ (denoted $\mathrm{tr}_{A/B,\sigma}$) to be the B-bilinear pairing given by the composition

$$\mathrm{tr}_{A/B,\sigma} : A \times A \xrightarrow{\mathrm{id}_A \times \sigma} A \times A \xrightarrow{\mu_A} A \xrightarrow{\mathrm{tr}} B.$$

That is, $\mathrm{tr}_{A/B,\sigma}(y, z) = \mathrm{tr}(y\sigma(z))$. In the case $\sigma = \mathrm{id}_A$, we use $\mathrm{tr}_{A/B}$ to denote $\mathrm{tr}_{A/B,\mathrm{id}_A}$.

Note that $\mathrm{tr}_{A/B,\sigma}(y, z) = \mathrm{tr}_{A/B,\sigma}(\sigma(z), \sigma^{-1}(y))$, so that this bilinear pairing need not be symmetric for a general σ, but it is symmetric if $\sigma = \mathrm{id}_A$.

Notation 3.3. Given a basis $\{x_1, \ldots, x_g\}$ of A as a B-module, the matrix of $\mathrm{tr}_{A/B,\sigma}$ with respect to this basis is $W_\sigma = (\mathrm{tr}(x_i\sigma(x_j)))_{ij}$. In this way, if y and z have representatives in the above basis given by vectors $\boldsymbol{y}$ and $\boldsymbol{z}$, then $\mathrm{tr}_{A/B,\sigma}(y, z) = \boldsymbol{y}^T W_\sigma \boldsymbol{z}$. We let W denote the matrix W_{id_A}.

Definition 3.4 ([2])**.** For a free extension A/B, we define the discriminant of A over B to be determinant of the matrix W representing the trace pairing $\mathrm{tr}_{A/B}$ with respect to some chosen basis of A over B, as in Notation 3.3.

Lemma 3.5. *Let A/B be a free extension of algebras, let $\sigma \in \mathrm{Aut}(A)$ restrict to the identity on B, and let $\{x_1, \ldots, x_g\}$ be a basis of B over A. Then using the same notation appearing in 3.1 and 3.3, one has $W_\sigma = WX_\sigma$ and therefore $\det(W_\sigma) = \det(W)\det_{A/B}\sigma$.*

Proof. Let y and z be elements of A, with representations $\boldsymbol{y}$ and $\boldsymbol{z}$ in the chosen basis, respectively. Then since $\mathrm{tr}_{A/B,\sigma} = \mathrm{tr}_{A/B} \circ(\mathrm{id}_A \times\sigma)$ one has the following string of equalities from which the claim follows:

$$\mathrm{tr}_{A/B,\sigma}(y, z) = \mathrm{tr}_{A/B}(y, \sigma(z)) = \boldsymbol{y}WX_\sigma\boldsymbol{z}.$$ □

4. Discriminants and reflections

In this section, we collect some results from classical commutative invariant theory that we will need for our examples. We prove that when G is a group generated by reflections acting on the polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$, the discriminant of the extension A/A^G may be computed in a manner similar to the formula for the discriminant of an algebraic number field, c.f. [12, Proposition 2.26]. Recall that our field $\Bbbk$ is of characteristic zero.

Let $\sigma \in \mathrm{GL}_n(\Bbbk)$. Recall that σ is a reflection if σ is of finite order and fixes a codimension one subspace of the vector space of linear forms in A. Denote the quotient field of a domain A by $Q(A)$.

Theorem 4.1. *Let $A = \Bbbk[x_1, \ldots, x_n]$ and $G \subseteq \mathrm{GL}_n(\Bbbk)$ a finite group generated by reflections that acts on A as automorphisms. Then:*

(1) The invariant ring $A^G = \Bbbk[f_1, \ldots, f_n]$ is a graded subalgebra of A, with the f_i algebraically independent.

(2) One has $\prod_i \deg(f_i) = |G|$.

(3) A is free as an A^G-module of rank $|G|$.

(4) $Q(A^G) = Q(A)^G$, where the action of G on $Q(A)$ is induced from the action of G on A.

Proof. The first claim is the Shephard-Todd-Chevalley theorem. The second claim is well-known, see [15, Corollary 4.4]. The third claim follows from a Hilbert series computation, and the last statement follows from considering the Galois extension $Q(A)/Q(A)^G$. $\square$

Lemma 4.2. *Let $A = \Bbbk[x_1, \ldots, x_n]$ and $G \subseteq \mathrm{GL}_n(\Bbbk)$ a finite group generated by reflections that acts on A as automorphisms. Then for all $f \in A$, one has*

$$\mathrm{tr}^{A/A^G}(f) = \mathrm{tr}^{Q(A)/Q(A^G)}(f) = \sum_{\sigma \in G} \sigma(f).$$

Proof. The extension $Q(A)/Q(A)^G = Q(A)/Q(A^G)$ is Galois, and hence we have that the usual trace map

$$\mathrm{tr}^{Q(A)/Q(A^G)} f = \sum_{\sigma \in G} \sigma(f) \in Q(A^G)$$

may be computed by the trace of the $Q(A^G)$-linear map $\theta_f^{Q(A)} : Q(A) \longrightarrow Q(A)$ given by multiplication by f. Since for all $f \in A$, one has $\theta_f^{Q(A)} = \theta_f^A \otimes_{A^G} Q(A^G)$, we have the desired result. $\square$

The following proposition is useful in computations involving discriminants of extensions of commutative polynomial rings, since in practice the matrix W from Notation 3.3 can be time-consuming to obtain directly. This result is reminiscent of the formula for the discriminant of an algebraic number field K in terms of the square of the determinant of the matrix whose entries correspond to the evaluations of an integral basis of $\mathcal{O}_K$ at the different embeddings of K into $\mathbb{C}$.

Proposition 4.3. *Let $A = \Bbbk[x_1, \ldots, x_n]$ and $G = \{\sigma_1, \ldots, \sigma_g\} \subseteq \mathrm{GL}_n(\Bbbk)$ a finite group generated by reflections. Let $\{z_1, \ldots, z_g\}$ be a basis of A as an A^G-module, W be the matrix of the trace form of the extension A/A^G with respect to this basis, and let M be the matrix $(\sigma_i(z_j))$. Then $W = M^T M$. As a consequence, one has $d(A/A^G) = (\det M)^2$.*

Proof. One has that

$$(M^T M)_{ij} = \sum_k \sigma_k(z_i)\sigma_k(z_j) = \sum_k \sigma_k(z_i z_j) = \mathrm{tr}(z_i z_j) = W_{ij}.$$

The claim regarding the discriminant of the extension A/A^G follows since $\det W = d(A/A^G)$. $\square$

We record a corollary of this proposition when G is generated by a single reflection for later use.

Corollary 4.4. *Let $A = \Bbbk[x_1, \ldots, x_n]$ and let σ be a reflection of order m. Let A^σ be the set of elements of A left invariant by σ, and let f be a linear form such that $\sigma(f) = \xi f$ for some primitive m^{th} root of unity ξ. Then $d(A/A^\sigma) =_{\Bbbk^\times} f^{(m-1)m}$.*

Proof. After a change of variable, we have $A = \Bbbk[f, y_2, \ldots, y_n]$ with $\sigma(f) = \xi f$ and $\sigma(y_i) = y_i$. Therefore, $A^\sigma = \Bbbk[f^m, y_2, \ldots, y_n]$ and hence a basis for A over A^σ is $\{1, \ldots, f^{m-1}\}$. The matrix M from the above proposition is therefore a Vandermonde matrix on the elements $\{f, \xi f, \xi^2 f, \ldots, \xi^{m-1} f\}$. Therefore by Proposition 4.3 we have that

$$\det W = \left(\prod_{i<j} (\xi^i f - \xi^j f) \right)^2 =_{\Bbbk^\times} f^{2\binom{m}{2}} = f^{(m-1)m}. \qquad \square$$

Corollary 4.5. *Let S be an algebra, $A = S[t]$, and $R = S[t^m]$, $m \in \mathbb{Z}_{>0}$. Then $d(A/R) =_{\Bbbk^\times} (t^{m-1})^m$.*

Proof. This follows by applying Corollary 4.4 to the automorphism σ of A defined by $\sigma(t) = \xi t$ where ξ is a primitive m^{th} root of unity. $\square$

Example 4.6. Let $A = \Bbbk[x_1, x_2, x_3]$ and $G = \mathcal{S}_3$, the symmetric group acting as permutations of x_i. A basis for A over A^G is $\{1, x_1, x_2, x_1^2, x_1x_2, x_1^2x_2\}$ (this is well-known, see e.g. [10, Proposition V.2.20]). Let $M = (\sigma_i(z_j))$ so that

$$M = \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_1^2x_2 \\ 1 & x_2 & x_1 & x_2^2 & x_1x_2 & x_1x_2^2 \\ 1 & x_3 & x_2 & x_3^2 & x_2x_3 & x_2x_3^2 \\ 1 & x_1 & x_3 & x_1^2 & x_1x_3 & x_1^2x_3 \\ 1 & x_2 & x_3 & x_2^2 & x_2x_3 & x_2^2x_3 \\ 1 & x_3 & x_1 & x_3^2 & x_1x_3 & x_1x_3^2 \end{pmatrix}.$$

It can be checked that the determinant of M is the cube of the Vandermonde determinant on the set $\{x_1, x_2, x_3\}$ and so it follows from Proposition 4.3 that

$$d(A/A^G) = \left[\prod_{i<j} (x_i - x_j) \right]^6.$$

In light of the Example 4.6, we conjecture that the following question has an affirmative answer.

Question 4.7. Suppose $G = S_n$ acts on $A = k[x_1, \ldots, x_n]$ as permutations. Is the discriminant of A over A^G the Vandermonde determinant on $\{x_1, \ldots, x_n\}$ to the power $n!$?

5. Discriminants of twisted tensor products

In this section, we prove the main theorem that allows us to calculate discriminants for certain Ore extensions and skew group algebras.

Theorem 5.1. *Let A be an algebra, M a submonoid of a group G, and suppose $\rho : G \longrightarrow \mathrm{Aut}(A)$ is a group homomorphism such that $\mathrm{im}(\rho|_M) \cap \mathrm{Inn}(A) = \{\mathrm{id}_A\}$ and $H := \ker \rho \cap M \subseteq C(M)$. Set $T = A \otimes_\tau \Bbbk[M]$, and suppose $R \subseteq C(T)$ is a subalgebra such that:*

(a) A is free over $A \cap R$ of rank $n < \infty$,

(b) $R = (A \cap R) \otimes k[H]$, and

(c) There exists a basis $\{x_1, \dots, x_\ell\}$ of $\Bbbk[M]$ over $\Bbbk[H]$ with $x_1 = e_M$ and $x_i \in M$ (c.f. Lemma 2.5).

Then:

$$d(T/R) = \Big(d(A/A\cap R)\Big)^\ell \ \Big(d(k[M]/k[H])\Big)^n.$$

Remark 5.2. Note that since $R \subseteq C(A \otimes_\tau \Bbbk[M])$ and $C(A \otimes_\tau \Bbbk[M]) = C(A)^M \otimes \Bbbk[H]$ by Lemma 2.2, one has $A \cap R = C(A)^M \cap R$.

Before giving the proof of Theorem 5.1, we need one more lemma:

Lemma 5.3. *Under the hypotheses of Theorem 5.1, for all $a \in A$ and $m \in M$, one has*

$$\mathrm{tr}(a \otimes m) = \begin{cases} \mathrm{tr}(a) \otimes \mathrm{tr}(m) & \textit{if } m \in H \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $B = A \cap R = C(A)^M \cap R$, and let $\{x_1, \dots, x_n\}$ be a basis of A over B. Then

$$\{x_i \otimes m_j \mid 1 \le i \le n \text{ and } 0 \le j \le \ell - 1\}$$

is a basis of T over R. Recall that for a fixed $1 \le \alpha \le n$ and $1 \le \beta \le n$, one has

$$(a \otimes m)(x_\alpha \otimes m_\beta) = a(mx_\alpha) \otimes mm_\beta. \tag{5.4}$$

Consider the coefficient $c_{\alpha\beta} \in R$ of $x_\alpha \otimes m_\beta$ when writing the product in (5.4) in the $\{x_i \otimes m_j\}$ basis. If $c_{\alpha\beta} \neq 0$, then hypothesis (b) implies that $c_{\alpha\beta}$ would have a summand of the form $a' \otimes m'$ for some $m' \in H$ such that $mm_\beta = m'm_\beta$. Since M is cancellative, one has $m \in H$.

For the case $m \in H$, write $ax_\alpha = \sum_{i=1}^n r_{i\alpha} x_\alpha$ for some $r_{i\alpha} \in A \cap R$ and $mm_\beta = \sum_{j=1}^\ell r'_{j\beta} m_\beta$ for some $r'_{j\beta} \in \Bbbk[H]$. Then one has

$$\begin{aligned} (a \otimes m)(x_\alpha \otimes m_\beta) &= ax_\alpha \otimes mm_\beta \\ &= \sum_{i=1}^n \sum_{j=1}^\ell (r_{i\alpha} \otimes r'_{j\beta})(x_i \otimes m_j). \end{aligned}$$

Therefore the trace of the map given by left multiplication of $a \otimes m$ satisfies

$$\mathrm{tr}(a \otimes m) = \sum_{\alpha=1}^n \sum_{\beta=1}^\ell (r_{\alpha\alpha} \otimes r'_{\beta\beta}) = \Big(\sum_{\alpha=1}^n r_{\alpha\alpha}\Big) \otimes \Big(\sum_{\beta=1}^\ell r'_{\beta\beta}\Big) = \mathrm{tr}(a) \otimes \mathrm{tr}(m). \qquad \square$$

Proof of Theorem 5.1. Using the same notation in the proof of Lemma 5.3, we have that

$$\{x_i \otimes m_j \mid 1 \le i \le n \text{ and } 0 \le j \le \ell - 1\}$$

is a basis of T over R. List this basis in the order

$$\{x_1 \otimes m_1, \dots, x_n \otimes m_1, x_1 \otimes m_2, \dots, x_n \otimes m_2, \dots, x_1 \otimes m_\ell, \dots, x_n \otimes m_\ell\}.$$

One may then think of the trace matrix corresponding to this ordered basis of T over R as a $m \times m$ block matrix with blocks of size $n \times n$, where the block is determined by index on the $\{m_j\}$ basis used.

Since $(x_i \otimes m_j)(x_{i'} \otimes m_{j'}) = x_i(m_j x_{i'}) \otimes m_j m_{j'}$, Lemma 5.3 shows that the $(j, j')^{\text{th}}$ block entry is the zero matrix whenever $m_j m_{j'} \notin H$. By Lemma 2.6 there is exactly one $m_{j'}$ such that $m_j m_{j'} \in H$. In addition, one has that when $m_j m_{j'} \in R \cap M$, then in the notation of Lemma 5.3 the $(j, j')^{\text{th}}$ block entry is $W_{m_j} \operatorname{tr}(m_j m_{j'})$. By Lemma 3.5 we have that the determinant of W_{m_j} is (up to a unit in B) the determinant of the matrix W of the trace form of A over B, which by definition is the discriminant $d(A/B)$.

Therefore up to a unit in B, the determinant of the matrix of the trace form of T over R is the determinant of the matrix that is the tensor product of the matrices of the trace forms of A over B and of $\Bbbk[M]$ over $\Bbbk[H]$. The claim now follows from the usual formula for the determinant of a tensor product of two matrices. $\square$

Example 5.5. Let $A = \Bbbk[x_1, \dots, x_n]$ and $\sigma \in \operatorname{Aut}(A)$ defined by $\sigma(x_1) = \xi x_1$, ξ a primitive sixth root of unity, and $\sigma(x_i) = x_i$ for $i = 2, \dots, n$. Then $A^\sigma = \Bbbk[x_1^6, x_2, \dots, x_n]$ and $d(A/A^\sigma) = x_1^{30}$ by Corollary 4.4.

Let $\rho : \mathbb{N} \to \operatorname{Aut}(A)$ be the monoid homomorphism sending 1 to σ as in Remark 2.1 and let M be the submonoid of $\mathbb{N}$ generated by $\{2, 3\}$. By restriction, $\rho : M \to \operatorname{Aut}(A)$ satisfies $\ker \rho = \{6k \mid k \in \mathbb{N}\} \subseteq C(M)$ and $\operatorname{im} \rho \cap \operatorname{Inn}(A) = \{\operatorname{id}_A\}$. Clearly, $\Bbbk[M] \cong \Bbbk[t^2, t^3]$ and $\Bbbk[H] \cong \Bbbk[t^6]$. A basis for $\Bbbk[t^2, t^3]$ as a module over $\Bbbk[t^6]$ is $\{1, t^2, t^3, t^4, t^5, t^7\}$ and a direct computation shows that $d(\Bbbk[M]/\Bbbk[H]) =_{\Bbbk^\times} t^{42}$.

Consider the twisted tensor product $T = A \otimes_\tau \Bbbk[t^2, t^3]$ and let $R = A^\sigma \otimes \Bbbk[t^6]$. By Theorem 5.1, we have

$$d(T/R) = d(A/A^\sigma)^6 (d(\Bbbk[M]/\Bbbk[H]))^6 =_{\Bbbk^\times} (x_1^{30} t^{42})^6.$$

6. Discriminants of Ore extensions

In this section, we apply Theorem 5.1 to the case of an Ore extension. Recall that by Remark 2.1, Ore extensions are a special case of the twisted tensor products studied in Sections 2 and 5.

Theorem 6.1. *Let A be an algebra and set $S = A[t; \sigma]$, where $\sigma \in \operatorname{Aut}(A)$ has order $m < \infty$ and no σ^i, $1 \le i < m$, is inner. Suppose $R \subseteq C(S)$ and set $B = R \cap A^\sigma$. If A is free over B of rank n, then*

$$d(S/R) =_{R^\times} (d(A/B))^m \left(t^{m-1}\right)^{mn}.$$

Proof. We claim that the hypotheses satisfy those of Theorem 5.1. We view $\mathbb{N}$ as a submonoid of the additive group of the integers and $\rho : \mathbb{Z} \to \operatorname{Aut}(A)$ as the group homomorphism sending 1 to σ. Then $S \cong A \otimes_\tau \Bbbk[\mathbb{N}]$. Since σ^i is not inner for any $i \neq km$, $k \in \mathbb{Z}$, then $\operatorname{im}(\rho \mid_{\mathbb{N}}) \cap \operatorname{Inn}(A) = \{\operatorname{id}_A\}$ and $\ker \rho = \{km \mid k \in \mathbb{Z}\}$.

Set $H = \ker \rho \cap \mathbb{N} = \{km \mid k \in \mathbb{N}\}$, then $\{1, \dots, m-1\}$ is a basis for $\Bbbk[\mathbb{N}]$ over $\Bbbk[H]$ implying Theorem 5.1 (c). The hypothesis that A is free over B of rank n is equivalent to Theorem 5.1 (a). By Lemma 2.2, $R = B[t^m] = (A \cap R) \otimes \Bbbk[H]$ Hence, Theorem 5.1 (b) is satisfied. The formula now follows from Theorem 5.1 and Corollary 4.5. □

Corollary 6.2. *Let $A = \Bbbk[x_1, \dots, x_n]$ and σ be a reflection of order k. Let f be a linear form that satisfies $\sigma(f) = \xi f$ where ξ is a primitive k^{th} root of unity. Then the discriminant of the Ore extension $A[t;\sigma]$ is (up to scalar) $f^{(k-1)k^2} t^{(k-1)k^2}$.*

Proof. This follows from applying Corollaries 4.4 and using the Ore extension discriminant formula from Theorem 6.1. □

As test cases, we consider Ore extensions of the ordinary polynomial ring, the (-1)-skew polynomial ring

$$V_n = \Bbbk_{-1}[x_1, \dots, x_n],$$

and the (-1)-skew Weyl algebra

$$W_n = \Bbbk\langle x_1, \dots, x_n \mid x_i x_j + x_j x_i = 1 \text{ for } i \neq j\rangle.$$

Note that $\mathrm{gr}(W_n) = V_n$.

Example 6.3 ([2, Example 1.7])**.** V_2 is the Ore extension $\Bbbk[x][y;\sigma]$ where $\sigma(x) = -x$ and $C(V_2) = \Bbbk[x^2, y^2]$. Clearly $\Bbbk[x]$ is free over $\Bbbk[x]^\sigma = \Bbbk[x^2]$ and $d(\Bbbk[x]/\Bbbk[x^2]) =_{\Bbbk^\times} x^2$ by Corollary 4.5. By Theorem 6.1,

$$d(V_2/C(V_2)) =_{\Bbbk^\times} (x^2)^2 (y)^4 = x^4 y^4.$$

Example 6.4. By [2, Lemma 4.1 (3)],

$$C(V_n) = \begin{cases} \Bbbk[x_1^2, \dots, x_n^2] & \text{if } n \text{ is even} \\ \Bbbk[x_1^2, \dots, x_n^2, \prod_i x_i] & \text{if } n \text{ is odd.} \end{cases}$$

Set $C_n = \Bbbk[x_1^2, \dots, x_n^2]$ regardless of whether n is even or odd. In either case, V_n is finitely generated free over C_n; this is proved in [2, Lemma 4.1 (4)] for n even but the proof applies equally well when n is odd. However, in the case n is odd we do not obtain useful information about the automorphism group of V_n because a given automorphism may not fix C_n. Regardless, we use Theorem 6.1 to inductively compute

$$d(V_n/C_n) =_{\Bbbk^\times} \left(\prod_{i=1}^{n} x_i^2 \right)^{2^{n-1}}. \tag{6.5}$$

This gives an alternate method for obtaining the discriminant in [2, Theorem 4.9 (1)].

The case $n = 2$ follows from Example 6.3. Suppose (6.5) holds for some n and set $S = V_n[x_{n+1};\sigma]$ where $\sigma(x_i) = -x_i$ for $i = 1, \dots, n$. If n is odd, then σ does not fix $\prod_i x_i$. Hence, $C(V_n) \cap V_n^\sigma = C_n$ in both cases when n is even or odd, and V_n is finitely generated free over C_n of rank 2^n. Thus, by Theorem 6.1,

$$d(S/C_{n+1}) = d(V_n/C_n)^2(2x_{n+1})^{2\cdot 2^n} =_{\Bbbk^\times} \left(\prod_{i=1}^{n+1} x_i^2\right)^{2^n}.$$

Example 6.6. Let $A = \Bbbk[x,y]$ and $\sigma \in \operatorname{Aut}(A)$ defined by $\sigma(x) = y$ and $\sigma(y) = x$. Let $S = A[t;\sigma]$.

We have $|\sigma| = 2$ and σ is not an inner automorphism. Since A is commutative, $C(A)^\sigma = A^\sigma = \Bbbk[x+y, xy]$. Thus $C(S) = A^\sigma[t^2]$. A basis for A over A^σ is $\{1, x\}$. An easy computation shows that

$$\operatorname{tr}(1) = 2, \quad \operatorname{tr}(x) = x + y, \quad \operatorname{tr}(x^2) = x^2 + y^2.$$

Thus, the trace matrix for A over A^σ is

$$\begin{pmatrix} 2 & x+y \\ x+y & x^2+y^2 \end{pmatrix}$$

and so $d(A/A^\sigma) = (x-y)^2$. By Theorem 6.1,

$$d(S/C(S)) =_{\Bbbk^\times} \left((x-y)^2\right)^2 \left(t^2\right)^2 = (x-y)^4 t^4.$$

The discriminant of $S/C(S)$ is not dominating in the sense of [2, Definition 2.1].

The discriminant computation above can also be seen by observing that $S \cong \Bbbk_{(p_{i,j})}[x_1, x_2, x_3]$ where $p_{2,3} = p_{3,2} = -1$ and all other $p_{i,j} = 1$. The isomorphism is given by $x_1 \leftrightarrow x + y$, $x_2 \leftrightarrow x - y$, and $x_3 \leftrightarrow t$. S is free over its center $C(S) = \Bbbk[x_1, x_2^2, x_3^2]$, and the discriminant (up to a constant) is $D = x_2^4 x_3^4$ [4, Proposition 2.8].

Question 6.7. If we instead take $\sigma \in \operatorname{Aut}(V_2)$ given by $\sigma(x) = -y$, $\sigma(y) = x$ and set $S = A[t;\sigma]$ so that S satisfies

$$xy = -yx, \quad tx = yt, \quad ty = -xt,$$

what is the discriminant $d(S/C(S))$?

Changing to generators that include the eigenvectors of σ does not give a skew-polynomial ring (as it did in the previous example). Because σ^2 is inner, Theorem 6.1 does not apply. In particular, $C(S) = \Bbbk[x^2 + y^2, x^2y^2, xyt^2, t^4]$ is not a UFD.

We are interested in the Ore extension $W_2[t;\sigma]$ with $\sigma(x) = y$ and $\sigma(y) = x$. Because $\operatorname{gr}(W_2) = V_2$, the discriminant $d(V_2/C(V_2)^\sigma)$ is a filtered version of the discriminant of $d(W_2/C(W_2)^\sigma)$. The Macaulay2 routines are not currently equipped to handle the computations for this discriminant. Instead, we pass to the homogenization of W_2.

Given $f \in \Bbbk\langle x_1, \dots, x_n\rangle$, define $\deg(f)$ by total degree with $\deg(x_i) \in \mathbb{Z}_+$ If $f = \sum_{i=0}^d f_k$ where f_k is the homogeneous component of f with $\deg(f_k) = k$, $f_d \neq 0$, then the homogenization of f by the central indeterminate t is then $H(f) = \sum f_k t^{d-k}$ where $d = \deg(f)$. It is clear that $H(f)$ is homogeneous.

Suppose A is an algebra generated by $\{x_1, \dots, x_n\}$ subject to the relations $r_1, \dots, r_m$ and such that $\deg(x_i) > 0$. The homogenization $H(A)$ of A is the algebra on the generators $\{t, x_1, \dots, x_n\}$ subject to the homogenized relations $H(r_i)$, $i = 1, \dots, m$, as well as the additional relations $tx_j - x_j t$, $1 \leq j \leq n$.

Theorem 6.8. *Suppose A is an algebra generated by $\{x_1, \dots, x_n\}$ subject to the relations $r_1, \dots, r_m$ and such that $\deg(x_i) > 0$. If A is finitely generated free over $R \subseteq C(A)$, then $H(A)$ is finitely generated free over $H(R)$ and*

$$d(H(A)/H(R)) =_{(H(R))^\times} H(d(A/R)).$$

Proof. Suppose A (and hence $H(A)$) is generated in degree 1. This is easily generalized to other cases. There is an isomorphism $H(A)[t^{-1}] \to A[t^{\pm 1}]$ fixing t and for $i = 1, \dots, n$, $x_i \mapsto t^{-1}x_i$. By [6, Lemma 1.3] and [2, Lemma 3.1]

$$d(H(A)[t^{\pm 1}]/R[t^{\pm 1}]) =_{(R[t^{\pm 1}])^\times} d(A[t]/R[t]) =_{(R[t])^\times} d(A/R).$$

Tracing back through the isomorphism and clearing fractions gives the result. $\square$

Example 6.9. Let $A = W_2$, the 2-dimensional (-1)-quantum Weyl algebra $A = \Bbbk\langle x, y \mid xy + yx = 1\rangle$. Note that $C(A) = \Bbbk[x^2, y^2]$. By [6, Theorem 0.1], $d(A/C(A)) =_{\Bbbk^\times} \left(4x^2y^2 - 1\right)^2$.

It follows from [9, Proposition 2.8] that $C(H(A)) = \Bbbk[x^2, y^2, t]$. Hence, by Theorem 6.8

$$d(H(A)/C(H(A))) =_{\Bbbk^\times} \left(4x^2y^2 - t^4\right)^2.$$

Example 6.10. Let A be as in the previous example and let σ be the automorphism $x \leftrightarrow y$. Then $\operatorname{gr}(A) = V_2$ and $C(A) = C(V_2) = \Bbbk[x^2, y^2]$. Moreover, $C(A)^\sigma = C(V_2)^\sigma = \Bbbk[x^2 + y^2, x^2y^2]$. Extend σ to $H = H(A)$ by $\sigma(t) = t$. Then $C(H)^\sigma = \Bbbk[x^2 + y^2, x^2y^2, t]$ so $\operatorname{rank}(A/C(A)^\sigma) = \operatorname{rank}(H/C(H)^\sigma) = 8$. Let $X = x^2 + y^2$, $Y = x^2y^2$, and $T = t$. Then

$$d(H/C(H)^\sigma) =_{\Bbbk^\times} (4Y - T^4)^4(X^2 - 4Y^4)^4.$$

By [4, Proposition 4.7],

$$d(A/C(A)^\sigma) =_{\Bbbk^\times} (4Y - 1)^4(X^2 - 4Y^4)^4, \text{ and}$$
$$d(V_2/C(V_2)^\sigma) =_{\Bbbk^\times} Y^4(X^2 - 4Y^4)^4.$$

7. Skew group algebras

We identify A with its image in $A\#G$ under the embedding $a \mapsto a \otimes e$.

Theorem 7.1. *Let A be an algebra and G a finite group that acts on A as automorphisms such that no non-identity automorphism is inner. Set $S = A\#G$ and identify A with its image under the embedding $a \mapsto a \otimes e$. Suppose A is free over $R \subseteq C(A)^G$. Then*

$$d(S/R) =_{R^\times} d(A/R)^{|G|}.$$

Proof. This follows almost immediately from Theorem 5.1. By hypothesis, there is a map $\rho : G \to \mathrm{Aut}(A)$, $\mathrm{im}\,\rho \cap \mathrm{Inn}(A) = \{\mathrm{id}_A\}$, and $H = \ker \rho = \{e_G\}$. Our hypotheses directly imply (a) and (b) in Theorem 5.1. Because the elements of G form a basis of $\Bbbk[G]$, we have $\ell = |G|$. $\square$

Example 7.2. Let $A = \Bbbk[x_1, x_2, x_3]$ and $G = \mathcal{S}_3$, the symmetric group acting as permutations of x_i. By Example 4.6,

$$d(A/A^G) = \left[\prod_{i<j}(x_i - x_j)\right]^6.$$

Set $S = A\#G$ and $R = A^G$ identified both in A and in $C(S)$. It follows from Theorem 7.1 that

$$d(S/R) =_{R^\times} \left[\prod_{i<j}(x_i - x_j)\right]^{36} \otimes e.$$

We are interested in the skew group algebra $V_n\#\mathcal{S}_n$ where $\mathcal{S}_n$ is the symmetric group on n letters acting as permutations on the x_i. We have that $C(V_n\#\mathcal{S}_n)$ may be identified with $C(V_n)^{\mathcal{S}_n}$. In the case when n is even we can describe this center explicitly.

Lemma 7.3. *Let $\mathcal{S}_n$ act on V_n as permutations of the variables and let $\mathrm{Inn}(V_n)$ denote the set of inner automorphisms induced by normal elements of V_n. Then $\mathcal{S}_n \cap \mathrm{Inn}(V_n) = \{e\}$.*

Proof. Let σ be a nontrivial permutation of $\{1, \dots, n\}$, and suppose that σ is an inner automorphism induced by the normal element $a \in V_n$. Choose i such that $\sigma(i) \neq i$. Then if one considers the equality $ax_i = x_{\sigma(i)}a$, one sees this is impossible since the set of monomials that appear on the left hand side is disjoint from the set of monomials which appear on the right hand side. $\square$

Lemma 7.4. *Let $E_n = \Bbbk[e_1, \dots, e_n]$ where the e_i are the elementary symmetric functions in the $x_1^2, \dots, x_n^2$. If n is even, then $C(V_n)^{S_n} = E_n$ and V_n is free over E_n of order $2^n n!$. Consequently, $V_n\#\mathcal{S}_n$ is finitely generated free over its center of order $2^n(n!)^2$.*

Proof. The elementary symmetric functions satisfy $\deg e_i = 2i$. Set $E_n = \Bbbk[e_1, \dots, e_n]$. We claim

$$\mathrm{rank}(V_n/E_n) = 2^n n!.$$

The Hilbert series of V_n is $H_{V_n}(t) = 1/(1-t)^n$ while that for E_n is

$$H_{E_n}(t) = \frac{1}{(1-t^2)(1-t^4)\cdots(1-t^{2n})}.$$

Let $H_n(t) = H_{V_n}(t)/H_{E_n}(t)$ and assume inductively that $H_n(1) = 2^n n!$. This clearly holds in the case $n = 1$. Thus,

$$H_{n+1}(t) = \frac{(1-t^2)(1-t^4)\cdots(1-t^{2(n+1)})}{(1-t)^{n+1}} = H_n(t) \cdot \frac{(1-t^{2(n+1)})}{1-t} = H_n \cdot (1+t^{n+1})(1+t+t^2+\cdots+t^n).$$

Hence, $H_{n+1}(1) = H_n(1) \cdot 2 \cdot (n+1) = 2^{n+1}(n+1)!$. Since $V_n\#\mathcal{S}_n$ has rank $n!$ over V_n it follows that it has rank $2^n(n!)^2$ over E_n.

Freeness follows from the Auslander-Buchsbaum formula. Since E_n is a polynomial ring then $\mathrm{pd}_{E_n}(V_n) = \mathrm{depth}_{E_n}(V_n) - \mathrm{depth}(E_n) = 0$.

That the center of $V_n \# \mathcal{S}_n$ is generated by the elementary symmetric functions follows from [2, Lemma 4.1 (3)] and Lemma 2.2 as no element of $\mathcal{S}_n$ acts as an inner automorphism by Lemma 7.3. $\square$

When n is odd the center of V_n is not a polynomial ring and it follows that $C(V_n \# \mathcal{S}_n)$ is also not a polynomial ring.

Example 7.5. Let $\mathcal{S}_2$ act on V_2 as above and set $S = V_2 \# \mathcal{S}_2$. Then $E_2 = C(S) = \Bbbk[X, Y]$ where $X = x^2 + y^2$ and $Y = x^2 y^2$. Since $d(V_2/E_2) = Y^2(X^2 - 4Y)^2$, then by Theorem 7.1, $d(S/C(S)) =_{\Bbbk^\times} [Y^2(X^2 - 4Y)^2]^2 \otimes e$.

8. Automorphism groups

In this section we apply our results on the discriminant to compute explicitly the automorphism groups in several cases.

8.1. **An Ore extension of $\Bbbk[x, y]$.** Let $A = \Bbbk[x, y]$ and $\sigma \in \mathrm{Aut}(A)$ defined by $\sigma(x) = y$ and $\sigma(y) = x$. Let $S = A[t; \sigma]$, so that S satisfies the relations

$$xy = yx, \quad tx = yt, \quad ty = xt.$$

By Example 6.6, $f := d(S/C(S)) = 16(x - y)^4 t^4$. Set $X = x + y$, $Y = xy$, and $T = t^2$, so that $f = 16(X^2 - 4Y)^2 T^2$. Any automorphism of S preserves the center and hence the discriminant up to scalar. Because the center is a UFD, we have that any automorphism either preserves the factors $(X^2 - 4Y)$ and T, or else it interchanges them (up to a scalar)

It follows that any automorphism g of A must take Y to either αY or βT, and T must go to either βT or αY for $\alpha, \beta \in \Bbbk^\times$. Hence the restriction of g to $\Bbbk_{-1}[Y, T]$ must be an automorphism of $\Bbbk_{-1}[Y, T]$. Since X is in the center of A, $g(X)$ must be in the center of A, and for g to map onto A we must have $g(X) = \gamma X + r$, where $r \in C(A) \cap \Bbbk_{-1}[Y, T] \subseteq C(A)$. All such maps (with $\alpha, \beta, \gamma \in \Bbbk^\times$) are automorphisms of A, so all automorphisms of A are triangular, in the sense of [2, Theorem 3(2)]. The automorphisms of A are (-1)-affine [4, Definition 1.7], but not affine.

Let $g \in \mathrm{Aut}(S)$ and suppose g preserves the factors (up to scalar multiple). Then $\deg(g(X^2)) \leq 2$ so $\deg(g(X)) = 1$. Similarly, $\deg(Y) = 2$ so $\deg(x) = \deg(y) = 1$. Moreover, $\deg(g(T)) = 2$ so $\deg(t) = 1$ and t is mapped to a scalar multiple of itself. Thus, we reduce to a linear algebra problem and conclude that all such g have the form,

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} d \\ d \\ 0 \end{pmatrix},$$

with $a, b, c \in \Bbbk^\times$ and $d \in \Bbbk$.

A similar argument follows in the case that g interchanges the factors. These automorphisms have the form

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} a & a & -b \\ a & a & b \\ -c & c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} d \\ d \\ 0 \end{pmatrix}$$

with $a, b, c \in \Bbbk^\times$ and $d \in \Bbbk$.

A is a skew-polynomial ring that satisfies H2, but not H1 of [4, p.12]. Compare with [4, Theorem 3.1]; $C(A) = \Bbbk[X, Y^2, Z^2]$ – note that $\alpha_1 = 1$ – and $\mathrm{Aut}(A)$ is not affine. We conjecture that $\mathrm{Aut}(A)$ is not tame – note that [4, Proposition 4.5] does not apply because $g(X)$ can contain a constant. (See the definitions of *elementary* and *tame* on p. 3 of [4].)

8.2. **An Ore extension of V_2.** Let $A = V_2$ with $\sigma \in \mathrm{Aut}(A)$ given by $\sigma(x) = y$, $\sigma(y) = x$. Set $S = A[t; \sigma]$ so that S satisfies

$$xy = -yx, \;\; tx = yt, \;\; ty = xt.$$

This example cannot be reduced to the skew polynomial case by using eigenvectors. Here A is not free over $A^\sigma = \Bbbk\langle x + y, x^3 + y^3\rangle$ and A^σ is not AS regular. However, A is free over the polynomial ring $C(A)^\sigma = \Bbbk[x^2 + y^2, x^2y^2]$, and $C(S) = \Bbbk[x^2 + y^2, x^2y^2, t^2] = \Bbbk[X, Y, T]$ is again a polynomial ring.

By Example 6.10 and Theorem 6.1,

$$d(S/C(S)) =_{\Bbbk^\times} T^8 Y^8 (X^2 - 4Y)^8.$$

The discriminant is not dominating, but the center is a UFD and we can eliminate cases easily.

Case 1: $g(X^2 - 4Y) = \alpha(X^2 - 4Y), g(Y) = \beta Y, g(T) = \gamma T$**.**

Then $g(X)^2 = g(X^2) = \alpha(X^2 - 4Y) + 4g(Y) = \alpha(X^2 - 4Y) + 4\beta Y = \alpha X^2 + 4(\beta - \alpha)Y$. For $\alpha X^2 + 4(\beta - \alpha)Y$ to be the square of some polynomial in $\Bbbk[X, Y, T]$ we need $\alpha = \beta$, and then $g(X) = \sqrt{\alpha}X$, $g(Y) = \alpha Y$. Then $g(Y) = g(x^2y^2) = -g((xy)^2) = -\alpha(xy)^2$ so that $g(xy)$ has degree 2 and so $g(x)$ and $g(y)$ have degree 1. If $g(T) = \gamma T$, then $g(t) = \sqrt{\gamma}t$. Hence the relations in A imply that all automorphisms of A are graded automorphisms and of the form

$$g(x) = ay, g(y) = ax, g(t) = bt \text{ or } g(x) = ax, g(y) = ay, g(t) = bt \text{ for } a, b \in \Bbbk.$$

The other cases are easily eliminated.

Case 2: $g(X^2 - 4Y) = \alpha(X^2 - 4Y), g(Y) = \beta T, g(T) = \gamma Y$**.**

$g(X)^2 = g(X^2) = \alpha(X^2 - 4Y) + 4\beta T$ cannot happen in $\Bbbk[X, Y, T]$.

Case 3: $g(X^2 - 4Y) = \alpha T, g(Y) = \beta Y, g(T) = \gamma(X^2 - 4Y)$**.** $g(X)^2 = g(X^2) = \alpha T + 4\beta Y$ cannot happen in $\Bbbk[X, Y, T]$.

Case 4: $g(X^2 - 4Y) = \alpha Y, g(Y) = \beta(X^2 - 4Y), g(T) = \gamma T$**.** $g(X)^2 = g(X^2) = \alpha Y + 4\beta(X^2 - 4Y)$ so $\alpha = 16\beta$ and $g(X) = 2\sqrt{\beta}X$ and $g(Y) = -g(xy)^2 = \beta(X^2 - 4Y)$ so $g(x)$ and $g(y)$ have degree 1; further $g(t) = \sqrt{\gamma}t$ and we are back to the automorphisms in Case 1.

Case 5: $g(X^2-4Y)=\alpha T, g(Y)=\beta(X^2-4Y), g(T)=\gamma Y$**.** $g(X)^2=g(X^2)=\alpha T+4\beta(X^2-4Y)$ cannot happen in $\Bbbk[X,Y,T]$.

Case 6: $g(X^2-4Y)=\alpha Y, g(Y)=\beta T, g(T)=\gamma(X^2-4Y)$**.** $g(X)^2=g(X^2)=\alpha Y+4\beta T$ cannot happen in $\Bbbk[X,Y,T]$. All automorphisms of S are graded.

Question 8.1. Let $R=\Bbbk_{-1}[x,y,z]$ and σ interchange x and y. Then $C(R[t;\sigma])=\Bbbk[x^2+y^2,x^2y^2,z^2,t^2]$ (σ eliminates xyz from $C(R)$). $\mathrm{Aut}(R)$ contains a free subgroup on two generators; does $\mathrm{Aut}(R[t;\sigma])$ also contain a free subgroup on two generators?

8.3. **An Ore extension of** W_2**.** Let $\sigma\in\mathrm{Aut}(W_2)$ be given by $\sigma(x)=y$ and $\sigma(y)=x$. Set $S=W_2[t;\sigma]$ so that S satisfies

$$xy+yx=1,\ \ tx=yt,\ \ ty=xt.$$

The center of S is $C(S)=\Bbbk[x^2+y^2,x^2y^2,t^2]$. Set $X=x^2+y^2$, $Y=x^2y^2$, and $T=t^2$. By Example 6.10 and Theorem 6.1, $d(S/C(S))=T^8(4Y-1)^8(X^2-4Y)^8$. Determining the automorphism group of S involves a case-by-case analysis that is nearly identical to that in 8.2.

Case 1: $g(X^2-4Y)=\alpha(X^2-4Y), g(4Y-1)=\beta(4Y-1), g(T)=\gamma T$**.** Then $g(X)^2=g(X^2)=\alpha(X^2-4Y)+g(4Y)=\alpha(X^2-4Y)+\beta(4Y-1)+1=\alpha X^2+4(\beta-\alpha)Y+(1-\beta)$. For $\alpha X^2+4(\beta-\alpha)Y$ to be the square of some polynomial in $\Bbbk[X,Y,Z]$ we need $\alpha=\beta=1$, and then $g(X)=X, g(Y)=Y$. Then $g(Y)=g(x^2y^2)=-g((xy)^2)=-(xy)^2$ so that $g(xy)$ has degree 2 and so $g(x)$ and $g(y)$ have degree 1. If $g(T)=\gamma T$, then $g(t)=\sqrt{\gamma}t$. Hence the relations in A imply that all automorphisms of A are affine automorphisms, and of the form

$$g(x)=ay, g(y)=a^{-1}x, g(t)=bt \text{ or } g(x)=ax, g(y)=a^{-1}y, g(t)=bt,$$

with $a=\pm1$. The other cases are easily eliminated.

Case 2: $g(X^2-4Y)=\alpha(X^2-4Y), g(4Y-1)=\beta T, g(T)=\gamma(4Y-1)$**.** $4Y-1$ has degree 4 and it follows that $g(4Y-1)$ has degree 4. Since T is of degree 2, we cannot have $g(4Y-1)=\beta T$.

Case 3: $g(X^2-4Y)=\alpha T, g(4Y-1)=\beta(4Y-1), g(T)=\gamma(X^2-4Y)$**.** $g(X)^2=g(X^2)=\alpha T+4\beta(4Y-1)$ is not a square in $\Bbbk[X,Y,Z]$.

Case 4: $g(X^2-4Y)=\alpha(4Y-1), g(4Y-1)=\beta(X^2-4Y), g(T)=\gamma T$**.** $g(X)^2=g(X^2)=\alpha(4Y-1)+4\beta(X^2-4Y)=2\beta X^2+4(\alpha-4\beta)Y-\alpha$. This is not a square in $\Bbbk[X,Y,T]$.

Case 5: $g(X^2-4Y)=\alpha T, g(4Y-1)=\beta(X^2-4Y), g(T)=\gamma(4Y-1)$**.** $g(X)^2=g(X^2)=4\alpha T+4\beta(X^2-4Y)$ is not a square in in $\Bbbk[X,Y,T]$.

Case 6: $g(X^2-4Y)=\alpha(4Y-1), g(4Y-1)=\beta T, g(T)=\gamma(X^2-4Y)$**.** See Case 2.

Hence, all automorphisms of W_2 are affine and of the form in Case 1.

8.4. **The homogenization of** W_2**.** Let $H=H(W_2)$ and $C=C(H)$. In Example 6.9 it was shown that $d(H/C)=_{\Bbbk^\times}\left(4x^2y^2-t^4\right)^2$.

Let I be a height one prime ideal of H. By [8, Theorem 6.6], either $I = (t)$, $I = (xy - yx)$, or $I = (g)$ with $\deg(g) > 1$. Given $\phi \in \mathrm{Aut}(H)$, it follows that $\phi(\deg(r)) \geq \deg(r)$ and so (t) is the only height one prime ideal generated by a degree one element. Hence $\phi(t) = \alpha t$ for some $\alpha \in \Bbbk^\times$. Thus, $\deg(\phi(t^2)) = 2$ and so $\deg(\phi(x^2y^2)) = 4$. We conclude that ϕ is affine.

Let $\phi \in \mathrm{Aut}(H)$ and write

$$\phi(x) = a_0 + a_1x + a_2y + a_3t, \quad \phi(y) = b_0 + b_1x + b_2y + b_3t, \quad \phi(t) = c_0 + c_1x + c_2y + c_3t,$$

with $a_i, b_i, c_i \in \Bbbk$ for $i = 0, \dots, 3$. Because t is central, then $c_1 = c_2 = 0$. Hence,

$$0 = \phi_0(xy + yx - t^2) = 2a_0b_0 - c_0^2$$

and

$$\begin{aligned}
0 &= \phi_1(xy + yx - t^2) \\
&= 2\left[a_0(b_1x + b_2y + b_3t) + b_0(a_1x + a_2y + a_3t) - c_0c_1t\right] \\
&= 2\left[(a_0b_1 + b_0a_1)x + (a_0b_2 + b_0a_2)y + (a_0b_3 + b_0a_3 - c_0c_1)t\right].
\end{aligned}$$

If $b_0 = 0$, then $c_0 = 0$ and $a_0b_1 = a_0b_2 = a_0b_3 = 0$. Since $\phi_1(y) \neq 0$, then $a_0 = 0$. Suppose $b_0 \neq 0$, then $-\frac{a_0}{b_0} = \frac{a_1}{b_1} = \frac{a_2}{b_2}$ so $a_1b_2 - a_2b_1 = 0$ and ϕ is not an isomorphism. Hence, we conclude that $a_0 = b_0 = c_0 = 0$. Thus,

$$\begin{aligned}
0 &= \phi(xy + yx - t^2) \\
&= 2(a_1b_1x^2 + a_2b_2y^2) + (a_1b_2 + a_2b_1)xy + (a_2b_1 + a_1b_2)yx - c_3^2t^2 \\
&= 2(a_1b_1x^2 + a_2b_2y^2) + (a_1b_2 + a_2b_1 - c_3^2)t^2.
\end{aligned}$$

We have two cases. Either $a_1 = b_2 = 0$ or $a_2 = b_1 = 0$ and c_3 is determined by the a_i, b_j. Thus,

$$\mathrm{Aut}(H) = (\Bbbk^\times)^2 \rtimes \{\tau\}$$

where τ is the involution interchanging x and y.

8.5. **The automorphism group of $V_2\#S_2$.** Set $A = V_2\#S_2$ and write $\mathcal{S}_2 = \{e, g\}$ as before. Example 7.5 shows that $d(A/C(A)) =_{\Bbbk^\times} [Y^2(X^2 - 4Y)^2]^2 \otimes e$ where $X = x^2 + y^2$ and $Y = x^2y^2$.

Because $C(A) \cong E_2$ is a PID, then any automorphism of A either preserves the factors Y and $X^2 - 4Y$ or else it interchanges them (up to a scalar). Suppose $\phi \in \mathrm{Aut}(V_2)$. It follows easily that $\deg(\phi(Y)) \leq 4$. If ϕ preserves the factors Y and $X^2 - 4Y$, then $\phi(Y) = k_1Y$ and $\phi(X^2 - 4Y) = k_2(X^2 - 4Y)$ for $k_1, k_2 \in \Bbbk^\times$. We have

$$k_2(X^2 - 4Y) = \phi(X^2 - 4Y) = \phi(X)^2 - 4k_1Y.$$

Thus, $\phi(X)^2 = k_2(X^2 - 4Y) - 4k_1Y$. As V_2 is a domain and the degree of the right-hand side is at most 4, then the degree of $\phi(X)$ is at most 2. A similar argument shows the same result when ϕ interchanges the factors.

Lemma 8.2. *Let $\phi \in \operatorname{Aut}(A)$, then $\phi(1 \otimes g) = \pm(1 \otimes g)$.*

Proof. Write $\phi(1 \otimes g) = a \otimes e + b \otimes g$. We have $(1 \otimes g)^2 = 1 \otimes e$, so

$$1 \otimes e = (a \otimes e + b \otimes g)^2 = (a^2 + b(g.b)) \otimes e + (ab + b(g.a)) \otimes g.$$

Hence, $a^2 + b(g.b) = 1$ and $ab + b(g.a) = 0$. Write $a = a_0 + a_1 + \cdots + a_d$ where $\deg(a_k) = k$ and similarly for b. We have $0 = (ab + b(g.a))_0 = 2a_0b_0$ and $1 = (a^2 + b(g.b))_0 = a_0^2 + b_0^2$. Thus, either $a_0 = \pm 1$ and $b_0 = 0$, or $b_0 = \pm 1$ and $a_0 = 0$.

Suppose $a_0 = 1$ and $b_0 = 0$. The remaining cases are similar. Then $0 = (a^2 + b(g.b))_1 = 2a_1$, so $a_1 = 0$, and $0 = (ab + b(g.a))_1 = 2a_0b_1$, so $b_1 = 0$.

We proceed by induction. Suppose $a_k = b_k = 0$ for all $k = 1, \ldots, n-1$. Then

$$0 = (a^2 + b(g.b))_n = \left((a_0 + a_n)^2\right)_n = 2a_0a_n,$$

so $a_n = 0$. Furthermore, $0 = (ab + b(g.a))_n = 2a_0b_n$, so $b_0 = 0$. □

Throughout, let $\phi \in \operatorname{Aut}(A)$ and write $\phi(x \otimes e) = r \otimes e + s \otimes g$.

As a consequence of the Lemma 8.2 we have

$$\phi(y \otimes e) = \phi((1 \otimes g)(x \otimes e)(1 \otimes g)) = (1 \otimes g)\phi(x \otimes e)(1 \otimes g).$$

Hence,

$$\phi(y \otimes e) = (1 \otimes g)(r \otimes e + s \otimes g)(1 \otimes g) = g.r \otimes e + g.s \otimes g.$$

Moreover, $x \otimes g = (x \otimes e)(1 \otimes g)$ and $y \otimes g = (y \otimes e)(1 \otimes g)$. Thus, the automorphism ϕ is completely determined by the choice of r and s.

Hence, we have the equations,

$$\phi((x^2 + y^2) \otimes e) = \left(r^2 + s(g.s) + g.r^2 + (g.s)s\right) \otimes e + (rs + s(g.r) + g.(rs) + (g.s)r) \otimes g, \tag{8.3}$$

$$\phi((xy + yx) \otimes e) = \left(r(g.r) + s^2 + (g.r)r + (g.s)^2\right) \otimes e + (r(g.s) + sr + (g.r)s + (g.s)(g.r)) \otimes g. \tag{8.4}$$

Lemma 8.5. *The degree zero components of r and s are zero.*

Proof. Since $xy + yx = 0$ $0 = \phi((xy + yx) \otimes e)$, then by restricting (8.4) to the degree zero component we find $r_0^2 + s_0^2 = 0$ and $r_0s_0 = 0$. The result now follows. □

Lemma 8.6. *Suppose* $\deg(r) > \deg(s) > 1$, *then* $s = 0$. *Similarly, if* $\deg(s) > \deg(r) > 1$, *then* $r = 0$.

Proof. Suppose $\deg(r) > \deg(s) > 1$ and write $r = r_1 + \cdots + r_d$ where $\deg(r_k) = k$ and by hypothesis $d > 1$. Because ϕ is an automorphism, then $\deg(\phi((x^2+y^2)\otimes e)) \leq 2$. By (8.3), $\left(r^2 + s(g.s) + g.r^2 + (g.s)s\right)_{2d} = 0$, then we have $r_d^2 + (g.r_d)^2 = 0$.

Because the action of g is diagonalizable, we can decompose r_d uniquely into a sum of elements from the two weight spaces, so $r_d = r_+ + r_-$ where $g.r_+ = r_+$ and $g.r_- = -r_-$. We then have

$$0 = r_d^2 + (g.r_d)^2 = 2(r_+^2 + r_-^2).$$

Because the weight spaces are disjoint, we conclude that $r_d = 0$.

A similar argument holds in the case $\deg(s) > \deg(r) > 1$ but we use (8.4) instead of (8.3). $\square$

Write $\hat{r}_k = r_k + g.r_k$ and $\hat{s}_k = s_k + g.s_k$ so that both $\hat{r}_k$ and $\hat{s}_k$ are fixed by the action of g and $\hat{r}_k = 0$ if and only if r_k belongs to the negative weight space. Since $(x+y)^2 = x^2 + y^2$ in V_2, then

$$\phi((x^2+y^2)\otimes e) = \phi\left((x+y)\otimes e\right)^2 = \left[\sum_{k=1}^{d} \hat{r}_k \otimes e + \hat{s}_k \otimes g\right]^2.$$

Let $\ell \in \{2, \ldots, d\}$ be the largest degree such that the above expression is nonzero. Then we have

$$\begin{aligned}
0 &= (\hat{r}_\ell \otimes e + \hat{s}_\ell \otimes g)^2 \\
&= \left(\hat{r}_\ell^2 + \hat{s}_\ell(g.\hat{s}_\ell)\right) \otimes e + (\hat{r}_\ell \hat{s}_\ell + \hat{s}_\ell(g.\hat{r}_\ell)) \otimes g \\
&= \left(\hat{r}_\ell^2 + \hat{s}_\ell^2\right) \otimes e + (\hat{r}_\ell \hat{s}_\ell + \hat{s}_\ell \hat{r}_\ell) \otimes g.
\end{aligned}$$

Each component must be zero and so $(\hat{r}_\ell + \hat{s}_\ell)^2 = 0$. Thus, $\hat{r}_\ell = -\hat{s}_\ell$ but because $\hat{r}_\ell^2 + \hat{s}_\ell^2 = 0$ then $\hat{r}_\ell = \hat{s}_\ell = 0$. Hence, all higher degree components of r and s are contained in the negative weight space.

Write $\phi((x+y)\otimes e) = u \otimes e + v \otimes g$ with $u, v \in V_2$. It follows from above that $d = \deg(u) = \deg(v)$ and u_k, v_k are contained in the negative weight space for $k > 1$. Then we have

$$\phi((x^2+y^2)\otimes e) = \phi\left([(x+y)\otimes e]^2\right) = [u \otimes e + v \otimes g]^2 = (u^2 - v^2) \otimes e + (uv - vu) \otimes g.$$

Assume $d > 1$. In the top degree we have $(u_d^2 - v_d^2) = 0$ and $(u_d v_d - v_d u_d) = 0$ so $(u_d - v_d)(u_d + v_d) = 0$. Hence, $u_d = \pm v_d$.

Case 1 ($u_d = v_d$): We claim $u_k = v_k$ for all $k \leq d$. Suppose this holds for some $\ell \leq d$.

$$\begin{aligned}
0 &= \left[u^2 - v^2\right]_{d+\ell-1} = \left[(u_1 + \cdots + u_d)^2 - (v_1 + \cdots + v_d)^2\right]_{d+\ell-1} \\
&= \left[(u_1 + \cdots + u_d)^2 - (v_1 + \cdots + v_{\ell-1} + u_\ell + \cdots + u_d)^2\right]_{d+\ell-1} \\
&= u_{\ell-1}u_d + u_d u_{\ell-1} - v_{\ell-1}u_d - u_d v_{\ell-1}.
\end{aligned}$$

$$\begin{aligned}
0 &= [uv - vu]_{d-\ell+1} = [(u_1 + \cdots u_d)(v_1 + \cdots + v_d) - (v_1 + \cdots + v_d)(u_1 + \cdots u_d)]_{d-\ell+1} \\
&= [(u_1 + \cdots u_d)(v_1 + \cdots + v_{\ell-1} + u_\ell + \cdots + u_d) - (v_1 + \cdots + v_{\ell-1} + u_\ell + \cdots + u_d)(u_1 + \cdots u_d)]_{d-\ell+1} \\
&= u_{\ell-1}u_d + u_d v_{\ell-1} - v_{\ell-1}u_d - u_d u_{\ell-1}.
\end{aligned}$$

Combining these gives

$$0 = u_d u_{\ell-1} - u_d v_{\ell-1} = u_d(u_{\ell-1} - v_{\ell-1}).$$

Hence, $u_{\ell-1} = v_{\ell-1}$.

Case 2 ($u_d = -v_d$): This case follows similarly to the above.

We conclude that $u_1 = \pm v_1$. An identical argument holds for $(x-y)\otimes e$. Thus, there exists $\alpha, \beta, \gamma, \delta \in \Bbbk$ such that,

$$\phi((x+y)\otimes e)_1 = \alpha(x+y)\otimes e + \gamma(x+y)\otimes g$$
$$\phi((x+y)\otimes g)_1 = \gamma(x+y)\otimes e + \alpha(x+y)\otimes g$$
$$\phi((x-y)\otimes e)_1 = \beta(x-y)\otimes e + \delta(x-y)\otimes g$$
$$\phi((x-y)\otimes g)_1 = \delta(x-y)\otimes e + \beta(x-y)\otimes g.$$

These elements generate the degree 1 component of V_2 and so the following matrix must be nonsingluar.

$$M = \begin{bmatrix} \alpha & 0 & \gamma & 0 \\ \gamma & 0 & \alpha & 0 \\ 0 & \beta & 0 & \delta \\ 0 & \delta & 0 & \beta \end{bmatrix}.$$

But $\det(M) = -(\beta^2 - \delta^2)(\alpha^2 - \gamma^2)$, a contradiction since the above argument gave us $\alpha = \pm\gamma$. Note that we assumed above that we are in the case that $\phi(1\otimes g) = 1\otimes g$ but the same argument works in the case $\phi(1\otimes g) = -1\otimes g$.

Write

$$\phi(x\otimes e) = (a(x+y) + b(x-y))\otimes e + (c(x+y) + d(x-y))\otimes g.$$

Because ϕ is an isomorphism and the image of $x\otimes e$ determines the isomorphism, then $a \neq \pm c$ and $b \neq \pm d$

Theorem 8.7. *Let $\phi \in V_2\#\mathcal{S}_2$ and write $\phi(x\otimes e) = (ax+by)\otimes e + (cx+dy)\otimes g$ for $a,b,c,d \in \Bbbk$. The parameters satisfy one of the three following conditions:*

- $a \in \Bbbk^\times$, $b = c = d = 0$;
- $b, d \in \Bbbk$, $b \neq 0$, $b \neq -d$, $a = -\frac{d^2}{b}$, $c = -d$;
- $c, d \in \Bbbk$, $c \neq -d$, $a = \pm\sqrt{\frac{c^2+d^2}{2}}$, $b = \mp\sqrt{\frac{c^2+d^2}{2}}$.

Proof. This is easily obtained by checking in Maple which parameters satisfy the defining relation and give a bijective map. ∎

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Wake Forest University, Department of Mathematics and Statistics, P. O. Box 7388, Winston-Salem, NC 27109