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The vanishing of a higher codimension analogue of Hochster's theta invariant

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Abstract We study H. Dao's invariant η_c^R of pairs of modules defined over a complete intersection ring R of codimension c having an isolated singularity. Our main result is that η_c^R vanishes for all pairs of modules when R is a *graded* complete intersection ring of codimension $c > 1$ having an isolated singularity. A consequence of this result is that all pairs of modules over such a ring are c-Tor-rigid.

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1 Introduction

Let R be an isolated complete intersection singularity, i.e., R is the quotient of a regular local ring $(Q, \mathfrak{m})$ by a regular sequence $f_1, \ldots, f_c \in Q$, and $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \neq \mathfrak{m}$. For any pair (M, N) of finitely generated R-modules, the Tor modules $\mathrm{Tor}_j^R(M, N)$ have finite length when $j \gg 0$. Moreover, the lengths of the odd and even indexed Tor modules in high degree follow predictable patterns. There are polynomials

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$$P_{\text{ev}}(j) = a_{c-1} j^{c-1} + \cdots + a_1 j + a_0 \text{ and } P_{\text{odd}}(j) = b_{c-1} j^{c-1} + \cdots + b_1 j + b_0$$

of degree at most $c - 1$ such that

$$\text{length}\operatorname{Tor}_{2j}^R(M, N) = P_{\text{ev}}(j) \text{ and } \text{length}\operatorname{Tor}_{2j+1}^R(M, N) = P_{\text{odd}}(j)$$

for all $j \gg 0$ (see, for example, [7, Theorem 4.1]).

The polynomials P_{ev} and P_{odd} need not be the same, nor is it necessary that the coefficients of j^{c-1} coincide. A natural invariant of the pair (M, N) is thus the difference, $a_{c-1} - b_{c-1}$, of these coefficients. Up to a constant factor, this difference is *Dao's η_c^R-invariant*.

For example, if $c = 1$, then $P_{\text{ev}} = a_0$ and $P_{\text{odd}} = b_0$ are both constants. This reflects the fact that, in our present context, the Eisenbud operator

$$\chi : \operatorname{Tor}_j^R(M, N) \xrightarrow{\cong} \operatorname{Tor}_{j-2}^R(M, N)$$

is an isomorphism for $j \gg 0$. In this case, the invariant $\eta_1^R(M, N)$ is (one half of) the difference $a_0 - b_0$ of these Tor-lengths. This difference is *Hochster's θ-invariant*.

In our previous paper [14] we studied Hochster's θ-invariant in the special case where R is a graded, isolated hypersurface singularity. Now we employ similar techniques to study the invariant η_c^R for graded, isolated complete intersection singularities of codimension $c > 1$. That is, we assume

$$R = \mathsf{k}[x_0, \ldots, x_{n+c-1}]/(f_1, \ldots, f_c) \tag{1.1}$$

where k is a field, the f_l's are homogeneous polynomials, and $\operatorname{Proj}(R)$ is a smooth k-variety. With these assumptions, the irrelevant maximal ideal $\mathfrak{m} = (x_0, \ldots, x_{n+c-1})$ of R is the only non-regular prime in $\operatorname{Spec} R$. Our main result is that, for such a ring R, the invariant $\eta_c^R(M, N)$ vanishes for all R-modules M and N, provided $c > 1$. See Theorem 4.5.

H. Dao [7, Theorem 6.3] has proven that the vanishing of $\eta_c^R(M, N)$ implies that the pair (M, N) is c-Tor-*rigid*, meaning that if c consecutive Tor modules vanish, then all subsequent Tor modules vanish too. We therefore conclude that all pairs of modules over rings of the form (1.1) having an isolated singularity are c-Tor-rigid, provided $c > 1$; our previous result [14, Remark 3.16] shows c-Tor-rigidity for $c = 1$ when the dimension of R is even. In general, one only has $(c + 1)$-Tor-rigidity for pairs of modules over a codimension c complete intersection [15].

We conjecture that $\eta_c^R(M, N) = 0$ for all pairs of modules (M, N) over an isolated complete intersection singularity of codimension $c > 1$, and hence that all pairs of modules over such a ring are c-Tor-rigid. There are many well-known examples (of isolated hypersurface singularities) where (M, N) is not 1-Tor-rigid, and hence $\theta^R(M, N) \neq 0$.

The invariant $\eta_c^R(M, N)$ is also defined for complete intersection rings that are not isolated singularities, provided the pair (M, N) has the property that $\operatorname{Tor}_j^R(M, N)$ has finite length for all $j \gg 0$. We include an example due to D. Jorgensen and O. Celikbas that shows that η_2^R need not vanish for a complete intersection of codimension 2 if the dimension of the singular locus is positive.

2 Dao's η_c^R-invariant

In this section we recall the definition of Dao's η_c^R-invariant for complete intersections.

Proposition 2.1 *Let R be the quotient of a noetherian ring Q by a regular sequence $f_1, \ldots, f_c$. For a pair of finitely generated R-modules M and N, suppose the Q-module $\operatorname{Tor}_j^Q(M, N)$*

vanishes for all $j \gg 0$ and that the R-modules $\operatorname{Tor}_j^R(M, N)$ are supported on a finite set of maximal ideals $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$ of R for all $j \gg 0$. Then there are polynomials $P_{\mathrm{ev}} = P_{\mathrm{ev}}^R(M, N)$ and $P_{\mathrm{odd}} = P_{\mathrm{odd}}^R(M, N)$ of degree at most $c - 1$ so that

$$\text{length} \operatorname{Tor}_{2j}^R(M, N) = P_{\mathrm{ev}}(j) \ \textit{and} \ \text{length} \operatorname{Tor}_{2j+1}^R(M, N) = P_{\mathrm{odd}}(j)$$

for all $j \gg 0$.

Proof Apply [7, Theorem 4.1(2)] to each $R_{\mathfrak{m}_i}$ and add the resulting polynomials to obtain the polynomials here. See Appendix A for an alternative proof of this result. □

The difference of the coefficients of j^{c-1} in $P_{\mathrm{ev}} = P_{\mathrm{ev}}^R(M, N)$ and in P_{odd} is the basis for an invariant of (M, N). We can obtain these coefficients through the $(c-1)$-st iterated first difference: the first difference of a polynomial $q(j)$ is the polynomial $q^{(1)}(j) = q(j) - q(j-1)$, and recursively one defines $q^{(i)} = (q^{(i-1)})^{(1)}$.

Definition 2.2 In the set up of Proposition 2.1, define

$$\eta_c^R(M, N) = \frac{(P_{\mathrm{ev}} - P_{\mathrm{odd}})^{(c-1)}}{2^c \cdot c!}.$$

This invariant of the pair (M, N) is Dao's η_c^R-invariant [7, 4.2].

Remark 2.3 For a pair of R-modules, Dao sets

$$\beta_j(M, N) = \begin{cases} \text{length} \operatorname{Tor}_j^R(M, N) & \text{if length} \operatorname{Tor}_j^R(M, N) < \infty \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown, as an easy application of Proposition 2.1, that under the assumptions in that result, if $c > 0$, then

$$\eta_c^R(M, N) = \lim_{n \to \infty} \frac{\sum_{j=0}^n (-1)^j \beta_j(M, N)}{n^c}.$$

This limit is the original definition of $\eta_c^R(M, N)$ due to Dao.

Our main result, Theorem 4.5, suggests the following conjecture:

Conjecture 2.4 *Suppose $R = Q/(f_1, \ldots, f_c)$ with Q a regular noetherian ring and $f_1, \ldots, f_c$ a regular sequence, with $c > 1$. If the singular locus of R consists of a finite number of maximal ideals, then $\eta_c^R(M, N) = 0$ for all finitely generated R-modules M and N.*

Remark 2.5 The case $N = \mathsf{k}$ of Conjecture 2.4 follows from a result due to L. Avramov, V. Gasharov, and I. Peeva [2, 8.1]. In this case, the length of the Tor_j record the Betti numbers of M over R, and part of their result states that both the even and odd Betti numbers grow at the same polynomial rate and have the same leading coefficient.

Example 2.6 The following example is due to D. Jorgensen [13, Example 4.1] and O. Celikbas [5, Example 3.11]. Let k be a field, and let $R = \mathsf{k}[\![x, y, z, u]\!]/(xy, zu)$. Then R is a local complete intersection of codimension two with positive dimensional singular locus. Let $M = R/(y, u)$, and let N be the cokernel of the map

$$R^2 \xrightarrow{\begin{pmatrix} 0 & u \\ -z & x \\ y & 0 \end{pmatrix}} R^3.$$

Then the pair (M, N) is not 2-Tor-rigid, and hence, by [7, 6.3], $\eta_2^R(M, N) \neq 0$.

3 The graded case

3.1 Let Q be a graded noetherian ring, $f_1, \ldots, f_c$ be a Q-regular sequence of homogeneous elements, and $R = Q/(f_1, \ldots, f_c)$. Then for each pair of finitely generated graded R-modules M and N, $\operatorname{Tor}_j^R(M, N)$ is a graded R-module for all j. Moreover, with the notation $d_l = \deg f_l$, the Eisenbud operators [8] $\chi_1, \ldots, \chi_c$ determine maps of graded R-modules

$$\chi_l \colon \operatorname{Tor}_j^R(M, N) \to \operatorname{Tor}_{j-2}^R(M, N)(-d_l)$$

for all j, where for a graded R-module T, we define $T(m)$ to be the graded R-module satisfying $T(m)_k = T_{k+m}$. Since the actions of the χ_l commute, we may view $\bigoplus_{j,i} \operatorname{Tor}_j^R(M, N)_i$ as a bigraded module over the bigraded ring $S = R[\chi_1, \ldots, \chi_c]$, where the degree of χ_l is $(-2, -d_l)$.

The operators χ_l first appeared in work of Gulliksen [10] as (co)homology operators (albeit in a different guise), where he proved that $\operatorname{Tor}_*^R(M, N)$ is artinian over S if and only if $\operatorname{Tor}_*^Q(M, N)$ is artinian over Q. Compare our Appendix A.

Proposition 3.2 *Let Q and R be as in paragraph 3.1. For a pair of finitely generated graded R-modules M and N, suppose the Q-module $\operatorname{Tor}_j^Q(M, N)$ vanishes for all $j \gg 0$ and there is a finite set of maximal ideals of R on which the R-modules $\operatorname{Tor}_j^R(M, N)$ are supported for all $j \gg 0$. Then the action of the Eisenbud operators induces an exact (Koszul) sequence*

$$\begin{aligned} &0 \to \operatorname{Tor}_j^R(M, N) \to \bigoplus_{l=1}^{c} \operatorname{Tor}_{j-2}^R(M, N)(-d_l) \to \\ &\bigoplus_{l_1<l_2} \operatorname{Tor}_{j-4}^R(M, N)(-d_{l_1} - d_{l_2}) \to \cdots \to \operatorname{Tor}_{j-2c}^R(M, N)(-d_1 - \cdots - d_c) \to 0 \end{aligned} \tag{3.1}$$

of graded R-modules for $j \gg 0$.

Proof Within this proof, we use a simplified grading on $S = R[\chi_1, \ldots, \chi_c]$, effectively ignoring the twists given by the d_l in (3.1). We let R lie in degree 0 and let each χ_l lie in degree -2. By [7, Lemma 3.2], for a sufficiently large J, the module $T = \bigoplus_{j \geq J} \operatorname{Tor}_j^R(M, N)$ is graded artinian over the ring S. Consider the Koszul complex $K = K[\chi_1, \ldots, \chi_c] \otimes_S T$. For $j \gg 0$, the complex (3.1) is the jth graded piece of K. We prove this complex K is exact in all but finitely many degrees.

As T is graded artinian over S, the total homology module $H(K)$ is as well. The descending chain of R-submodules

$$H(K) \supseteq \bigoplus_{j \geq 1} H(K)_j \supseteq \bigoplus_{j \geq 2} H(K)_j \supseteq \cdots$$

intersects to 0. Since $\chi_l H(K) = 0$, see [16, IV.A.4], these R-submodules are in fact S-submodules. Since $H(K)$ is artinian over S, the descending chain stabilizes. Thus there exists an $m > 0$ such that $H(K)_j = 0$ for all $j \geq m$. □

3.3 For the remainder of this section, we assume $Q = \mathsf{k}[x_0, \ldots, x_{n+c-1}]$ is a polynomial ring over a field with each x_i of degree one, $f_1, \ldots, f_c$ is a Q-regular sequence of homogeneous elements, and $R = Q/(f_1, \ldots, f_c)$. Let $d_l = \deg(f_l)$. In particular, R is graded. When M and N are finitely generated graded R-modules, the torsion modules $\operatorname{Tor}_j^R(M, N)$ are also graded.

Definition 3.4 Let R be as in paragraph 3.3. For finitely generated graded R-modules M and N, and an integer F, define

$$G_F(x,t) = \sum_{i,j\geq 0} \dim_{\mathsf{k}} \left(\operatorname{Tor}^R_{F+2j}(M,N)_i\right) t^i x^j \in \mathbb{Q}[[x,t]].$$

Remark 3.5 Note that if for some $F \gg 0$, $\operatorname{Tor}^R_{F+2j}(M,N)$ has finite length for all $j \geq 0$, then $G_F(x,t)$ belongs to $(\mathbb{Q}[t])[[x]]$.

For a finitely generated graded R-module T, its *Hilbert series* is

$$H_T(t) = \sum_{i\geq 0} \dim_{\mathsf{k}}(T_i)t^i.$$

$H_T(t)$ is a rational function with a pole of order $\dim T$ at $t = 1$. In fact,

$$H_T(t) = \frac{e_T(t)}{(1-t)^{\dim T}}, \tag{3.2}$$

where $e_T(t)$ is a Laurent polynomial [1, (1.1)], sometimes called the *multiplicity polynomial* of T. The multiplicity polynomial of R is calculated by using the presentation $R = Q/(f_1,\dots,f_c)$; explicitly, since $\deg(x_i) = 1$ and $\deg(f_l) = d_l$,

$$e_R(t) = \prod_{l=1}^{c}(1-t^{d_l})/(1-t)^c. \tag{3.3}$$

For graded R-modules M and N, we write the Hilbert series of $\operatorname{Tor}^R_j(M,N)$ as $H_j(t)$ or just H_j. If M and N are such that the $\operatorname{Tor}^R_j(M,N)$ have finite length for $j \gg 0$, then $H_j(t)$, $j \geq 0$, has the property that the number of initial terms of $H_j(t)$ that vanish goes to infinity as $j \to \infty$. It thus makes sense to form the sum

$$\sum_{j\geq 0}(-1)^j H_j(t),$$

and, more generally, to evaluate $G_F(x,t)$ at $x = 1$ (or any constant; see Remark 3.5). Observe that

$$\sum_{j\geq 0}(-1)^j H_{j+F}(t) = G_F(1,t) - G_{F+1}(1,t), \tag{3.4}$$

for any F.

Lemma 3.6 *Let R be a graded ring as in paragraph 3.3, and let M, N be finitely generated graded R-modules such that $\operatorname{Tor}^R_j(M,N)$ has finite length for $j \gg 0$. For $F \gg 0$ there is a unique polynomial $b_F(x,t) \in \mathbb{Q}[x,t]$ such that*

$$G_F(x,t) = \frac{b_F(x,t)}{\prod_{l=1}^{c}(1-t^{d_l}x)}. \tag{3.5}$$

For $E \gg 0$ and even there is a unique polynomial $\eta^R_{c,E}(M,N)(t) \in \mathbb{Q}[t]$ such that

$$\sum_{j\geq E}(-1)^j H_j(t) = \frac{\eta^R_{c,E}(M,N)(t)}{e_R(t)(1-t)^c} \quad \text{and} \quad \eta^R_{c,E}(M,N)(1) = 2^c\cdot c!\cdot \eta^R_c(M,N), \tag{3.6}$$

where $e_R(t)$ is the multiplicity polynomial of R defined in (3.3).

Proof Let $s_0 = 1$, $s_1 = t^{d_1} + \cdots + t^{d_c}$, ..., and $s_c = t^{d_1+\cdots+d_c}$; that is, the s_k are elementary symmetric functions on the symbols t^{d_l}. The exactness of (3.1) for $j \gg 0$ gives the relation

$$s_0 H_j - s_1 H_{j-2} + s_2 H_{j-4} + \cdots + (-1)^c s_c H_{j-2c} = 0,$$

from which it follows that, for $F \gg 0$, we have

$$(1 - s_1 x + s_2 x^2 - \cdots + (-1)^c s_c x^c) G_F(x,t) = b_{0,F}(t) + b_{1,F}(t)x + \cdots + b_{c-1,F}(t)x^{c-1},$$

for some polynomials $b_{i,F}(t)$. Set $b_F(x,t)$ equal to the right hand side of this equation and use $\sum_{k=0}^{c}(-1)^k s_k x^k = \prod_{l=1}^{c}(1 - t^{d_l}x)$. Then (3.5) follows easily.

To establish (3.6), observe that for $F \gg 0$, we have

$$\frac{b_F(x,1)}{(1-x)^c} = G_F(x,1) = \sum_{j \geq 0} \dim_{\mathsf{k}} \operatorname{Tor}^R_{F+2j}(M,N)x^j.$$

Taking $E \gg 0$ to be even, set $\eta^R_{c,E}(M,N)(t)$ to be $b_E(1,t) - b_{E+1}(1,t)$. The first equation in (3.6) follows immediately from (3.3), (3.4), and (3.5).

The leading coefficients of $P_{\mathrm{ev}}(M,N)$ and $P_{\mathrm{odd}}(M,N)$ are $b_E(1,1)/(c-1)!$ and $b_{E+1}(1,1)/(c-1)!$, and so the value of $\eta^R_{c,E}(M,N)(t)$ at $t=1$ is $2^c \cdot c! \cdot \eta^R_c(M,N)$. □

Remark 3.7 The polynomials $\eta^R_{c,E}(M,N)(t)$ depend on E, but, as Lemma 3.6 shows, they have a common value at $t=1$.

4 The vanishing of η^R_c

Throughout the remainder of this paper, we use the following notations and assumptions:

- k is a field;
- $R = \mathsf{k}[x_0, \ldots, x_{n+c-1}]/(f_1, \ldots, f_c)$, where $\deg x_i = 1$ for all i and the f_l's are homogeneous polynomials in $\mathfrak{m} = (x_0, \ldots, x_{n+c-1})$ with $d_l = \deg(f_l)$; (4.1)
- $c > 0$ and $f_1, \ldots, f_c$ forms a regular sequence;
- $X = \mathrm{Proj}(R)$ is a smooth k-variety.

Remark 4.1 Recall that the variety X is smooth if and only if $\mathfrak{m}$ is the radical of the homogeneous ideal generated by the f_l's and the maximal minors of the Jacobian matrix $(\partial f_l/\partial x_i)$. In particular, by the smoothness assumption, $\mathfrak{m} = (x_0, \ldots, x_{n+c-1})$ is the only non-regular prime of R.

For a quasi-projective scheme Z over a field k, we let $G(Z)$ and $K(Z)$ denote the Grothendieck groups of coherent sheaves and locally free coherent sheaves, respectively. Recall that if Z is regular (for example, if it is smooth over k), then the canonical map $K(Z) \to G(Z)$ is an isomorphism. For further explanation and discussion of the relevant groups and maps, see [14, §2.1].

From the assumptions (4.1), the smooth variety $X = \mathrm{Proj}(R) \subset \mathbb{P}^{n+c-1}$ has dimension $n-1$ and degree $d = d_1 \cdots d_c$. When k is infinite, there is a linear rational map $\mathbb{P}^{n+c-1} \dashrightarrow \mathbb{P}^{n-1}$ that determines a regular function on an open subset containing X, and hence it induces a regular map

$$\rho \colon X \to \mathbb{P}^{n-1}$$

that is finite, flat, and of degree d. As X and $\mathbb{P}^{n-1}$ are smooth and ρ is finite, we have the following map and isomorphisms:

$$\rho_*\colon K(X) \cong G(X) \to G(\mathbb{P}^{n-1}) \cong K(\mathbb{P}^{n-1}).$$

We also have the isomorphism

$$\mathbb{Z}[t]/(1-t)^n \cong K(\mathbb{P}^{n-1})$$

that sends $t \mapsto [\mathcal{O}(-1)]$. We will often identify $K(\mathbb{P}^{n-1})$ with $\mathbb{Z}[t]/(1-t)^n$; for example, if $\alpha \in K(X)$, then $\rho_*(\alpha)$ is interpreted as belonging to $\mathbb{Z}[t]/(1-t)^n$. Likewise, we identify $K(\mathbb{P}^{n-1}) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\mathbb{Q}[t]/(1-t)^n$.

The following three results are similar to [14, 4.1, 4.2, and 4.3]. We will establish the vanishing of $\eta_c^R(M,N)$ for $c > 1$ by employing them in the same way that we used our earlier results to show the vanishing of $\theta^R(M,N) = 2\eta_1^R(M,N)$ when $\dim R$ is even [14, Theorem 3.2].

Lemma 4.2 [14, Lemma 4.1] *Under the assumptions in (4.1) with* k *infinite, let* M *be a finitely generated graded* R*-module and* $\widetilde{M}$ *the associated coherent sheaf on* X*. Then*

$$\rho_*([\widetilde{M}]) = (1-t)^n H_M(t)$$

in $K(\mathbb{P}^{n-1})_{\mathbb{Q}} = \mathbb{Q}[t]/(1-t)^n$*. In particular,*

$$\rho_*(1) = \rho_*([\mathcal{O}_X]) = e_R(t) = \prod_{i=1}^{c}(1+t+t^2+\cdots+t^{d_i-1}) \in \mathbb{Q}[t]/(1-t)^n.$$

Proof The proof of [14, Lemma 4.1], which is the $c=1$ case, applies verbatim. □

Lemma 4.3 [14, Lemma 4.2] *Under the assumptions in (4.1) with* k *infinite, let* M *and* N *be finitely generated graded* R*-modules. For any sufficiently large even integer* E *and for any integer* $m \geq 0$*, the rational function*

$$(1-t)^{n+m-c}\frac{\eta_{c,E}^R(M,N)(t)}{(e_R(t))^2}$$

does not have a pole at $t=1$*. Its image in* $\mathbb{Q}[t]_{(t)}/(1-t)^n = \mathbb{Q}[t]/(1-t)^n = K(\mathbb{P}^{n-1})_{\mathbb{Q}}$ *satisfies the equation*

$$(1-t)^{n+m-c}\frac{\eta_{c,E}^R(M,N)(t)}{(e_R(t))^2} = (1-t)^m\left(\frac{\rho_*([\widetilde{M}])}{\rho_*(1)}\cdot\frac{\rho_*([\widetilde{N}])}{\rho_*(1)} - \frac{\rho_*([\widetilde{M}]\cdot[\widetilde{N}])}{\rho_*(1)}\right).$$

In particular, taking $m = c-1$ *yields*

$$(1-t)^{n-1}\eta_c^R(M,N) = \frac{(1-t)^{c-1}}{2^c\cdot c!}\left(\frac{d\cdot\rho_*([\widetilde{M}])}{\rho_*(1)}\cdot\frac{d\cdot\rho_*([\widetilde{N}])}{\rho_*(1)} - \frac{d^2\cdot\rho_*([\widetilde{M}]\cdot[\widetilde{N}])}{\rho_*(1)}\right),$$

where $\deg X = d = d_1\cdots d_c$*.*

Proof As in [14, Lemma 4.2], start with the equation of Hilbert series from [1, Lemma 7], namely

$$\frac{H_M H_N}{H_R} = \sum_{j\geq 0}(-1)^j H_j.$$

Splitting the sum at $E \gg 0$ and using the first relation in (3.6) gives

$$\frac{H_M \cdot H_N}{H_R} = \sum_{j=0}^{E-1} (-1)^j H_j + \frac{\eta^R_{c,E}(M,N)(t)}{e_R(t) \cdot (1-t)^c}.$$

Upon multiplying by $(1-t)^m/H_R = (1-t)^{n+m}/e_R(t)$ and rearranging, this yields

$$\begin{aligned} (1-t)^m \frac{(1-t)^n H_M}{e_R(t)} \cdot \frac{(1-t)^n H_N}{e_R(t)} - (1-t)^m \sum_{j=0}^{E-1} (-1)^j \frac{(1-t)^n H_j}{e_R(t)} \\ = \frac{\eta^R_{c,E}(M,N)(t)}{(e_R(t))^2} (1-t)^{n+m-c}. \end{aligned} \tag{4.2}$$

The first assertion follows from the fact that the expression before the equality in (4.2) does not have a pole at $t = 1$, using (3.2). Since both sides of this equation are power series in powers of $1 - t$, we may take their images in $\mathbb{Q}[t]/(1-t)^n$. Apply Lemma 4.2. The expression before the equality in (4.2) becomes

$$(1-t)^m \frac{\rho_*([\widetilde{M}])}{\rho_*(1)} \frac{\rho_*([\widetilde{N}])}{\rho_*(1)} - (1-t)^m \sum_{j=0}^{E-1} (-1)^j \frac{\rho_*([\widetilde{\mathrm{Tor}^R_j(M,N)}])}{\rho_*(1)}.$$

If E is large enough so that $\mathrm{Tor}^R_j(M,N)$ has finite length for $j \geq E$, then the alternating sum is $\rho_*([\widetilde{M}] \cdot [\widetilde{N}])/\rho_*(1)$ where $[\widetilde{M}] \cdot [\widetilde{N}]$ is multiplication in the ring $K(X)$. This gives the first equation in the Lemma.

For the second equation, set $m = c - 1$. Define $g(t) = \eta^R_{c,E}(M,N)(t)/(e_R(t))^2$. As $e_R(1) = d$ and $\eta^R_{c,E}(M,N)(t)$ is a polynomial, $g(t)$ is a rational function without a pole at $t = 1$. Thus, modulo $(1-t)^n$,

$$g(t)(1-t)^{n-1} = g(1)(1-t)^{n-1} + \frac{g(t)-g(1)}{1-t}(1-t)^n \equiv g(1)(1-t)^{n-1}.$$

Multiplication by d^2 establishes the second equation. □

In the next lemma and in the proof of our main theorem, we use étale cohomology. Assume k is a separably closed field, fix a prime $\ell \neq \operatorname{char} \mathsf{k}$, and write $\mathrm{H}^j_{\acute{e}t}(Z, \mathbb{Q}_\ell(i))$ for the étale cohomology of a scheme Z with coefficients in $\mathbb{Q}_\ell(i)$. In addition, write $\mathrm{H}^{2*}_{\acute{e}t}(Z, \mathbb{Q}_\ell(*))$ for $\bigoplus_i \mathrm{H}^{2i}_{\acute{e}t}(Z, \mathbb{Q}_\ell(i))$. This is a commutative ring under cup product. Moreover, the étale Chern character gives a ring homomorphism

$$ch_{\text{ét}}\colon K(Z)_{\mathbb{Q}} \to \mathrm{H}^{2*}_{\text{ét}}(Z, \mathbb{Q}_\ell(*)).$$

We refer the reader to [9] for additional background.

Lemma 4.4 [14, Lemma 4.3] *Under the assumptions in (4.1) with* k *separably closed, the diagram*

$$\begin{tikzcd} K(X)_{\mathbb{Q}} \arrow[r, "\frac{d}{\rho_*(1)} \cdot \rho_*"] \arrow[d, "ch_{\text{ét}}"'] & K(\mathbb{P}^{n-1})_{\mathbb{Q}} \arrow[d, "ch_{\text{ét}}"] \\ \mathrm{H}^{2*}_{\text{ét}}(X, \mathbb{Q}_\ell(*)) \arrow[r, "\rho_*^{\text{ét}}"'] & \mathrm{H}^{2*}_{\text{ét}}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*)) \end{tikzcd} \tag{4.3}$$

commutes, where $\rho_*^{\text{ét}}$ *is the push-forward map for étale cohomology.*

Proof The proof of [14, Lemma 4.4], which is the $c = 1$ case, applies verbatim. □

The following is the main result of this paper.

Theorem 4.5 *Under the assumptions in (4.1) with* k *separably closed, let* M *and* N *be finitely generated graded* R*-modules. For* E *a sufficiently large even integer,* $\eta^R_{c,E}(M,N)(t)$ *has a zero at* $t = 1$ *of order at least* $c - 1$*. In particular,* $\eta^R_c(M,N) = 0$ *for* $c > 1$.

Proof The claim is vacuous if $c \le 1$, and so assume $c > 1$. Let γ be the element $ch_{\text{ét}}(1-t)$ of $\mathrm{H}^{2*}_{\text{ét}}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*))$.

We apply $ch_{\text{ét}}$ to d^2 times the first equation in Lemma 4.3 and use the commutative diagram in Lemma 4.4 to obtain

$$\frac{d^2 \cdot ch_{\text{ét}}\left((1-t)^{n+m-c} \cdot \eta^R_{c,E}(M,N)(t)\right)}{ch_{\text{ét}}((e_R(t))^2)} = \\ \gamma^m \left(\rho_*^{\text{ét}} ch_{\text{ét}}([\widetilde{M}]) \cdot \rho_*^{\text{ét}} ch_{\text{ét}}([\widetilde{N}]) - d \cdot \rho_*^{\text{ét}}\left(ch_{\text{ét}}([\widetilde{M}]) \cdot ch_{\text{ét}}([\widetilde{N}])\right)\right). \tag{4.4}$$

For $\alpha, \beta \in \mathrm{H}^{2*}_{\text{ét}}(X, \mathbb{Q}_\ell(*))$, define

$$\Psi_m(\alpha,\beta) = \gamma^m\left(\rho_*^{\text{ét}}(\alpha) \cdot \rho_*^{\text{ét}}(\beta) - d \cdot \rho_*^{\text{ét}}(\alpha \cdot \beta)\right).$$

We will prove that Ψ_m vanishes for any $m \ge 1$. Using the projection formula

$$\rho_*^{\text{ét}}(\rho^*_{\text{ét}}(\alpha') \cdot \omega) = \alpha' \rho_*^{\text{ét}}(\omega)$$

with $\omega = 1$, and using the fact that $\rho_*^{\text{ét}}(1) = d$, we see that if $\alpha = \rho^*_{\text{ét}}(\alpha')$ for some $\alpha' \in \mathrm{H}^{2*}_{\text{ét}}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*))$, then

$$\Psi_m(\alpha,\beta) = \gamma^m(\alpha' \rho_*^{\text{ét}}(1)\rho_*^{\text{ét}}(\beta) - d \cdot \alpha' \rho_*^{\text{ét}}(\beta)) = 0.$$

Likewise, $\Psi_m(\alpha,\beta) = 0$ if $\beta = \rho^*_{\text{ét}}(\beta')$. But since X is a complete intersection in projective space, the map

$$\rho^*_{\text{ét}}: \mathrm{H}^{2j}_{\text{ét}}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(j)) \to \mathrm{H}^{2j}_{\text{ét}}(X, \mathbb{Q}_\ell(j))$$

is an isomorphism in all degrees except possibly in degree $2j = n-1$ (see [18, XI.1.6]). So we may assume n is odd and that $\alpha, \beta \in \mathrm{H}^{n-1}_{\acute{e}t}(X, \mathbb{Q}_\ell(\frac{n-1}{2}))$. Noticing that γ is in $\bigoplus_{i \geq 1} \mathrm{H}^{2i}_{\acute{e}t}(X, \mathbb{Q}_\ell(i))$, we see that $\gamma^m \rho^{\acute{e}t}_*(\alpha) \cdot \rho^{\acute{e}t}_*(\beta)$ and $\gamma^m d \cdot \rho^{\acute{e}t}_*(\alpha \cdot \beta)$ belong to

$$\bigoplus_{j \geq 0} \mathrm{H}^{2n-2+2m+2j}_{\acute{e}t}(X, \mathbb{Q}_\ell(n-1+m+j)).$$

This group vanishes when $m \geq 1$ because $\dim(X) = n-1$ and étale cohomology vanishes in degrees more than twice the dimension of a smooth variety over a separably closed field [17, X.4.3]. As Ψ_m is zero for $m \geq 1$, so too is the expression on the left-hand side of (4.4).

We have proven that for $m \geq 1$,

$$\frac{d^2 \cdot ch_{\acute{e}t}((1-t)^{n+m-c} \cdot \eta^R_{c,E}(M, N)(t))}{ch_{\acute{e}t}(e_R(t))^2} = 0$$

and hence

$$ch_{\acute{e}t}((1-t)^{n+m-c} \cdot \eta^R_{c,E}(M, N)(t)) = 0.$$

But the Chern character map with $\mathbb{Q}_\ell$ coefficients induces an isomorphism on projective space,

$$ch_{\acute{e}t} : K(\mathbb{P}^{n-1}) \otimes \mathbb{Q}_\ell \xrightarrow{\cong} \mathrm{H}^{2*}_{\acute{e}t}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(*)),$$

and therefore, $(1-t)^{n+m-c} \cdot \eta^R_{c,E}(M, N)(t) = 0$ in $\mathbb{Q}[t]/(1-t)^n$. That is, $(1-t)^n$ divides $(1-t)^{n+m-c} \cdot \eta^R_{c,E}(M, N)(t)$ in $\mathbb{Q}[t]$ so that $\eta^R_{c,E}(M, N)(t)/(1-t)^{c-m}$ is a polynomial. Taking $m = 1 < c$ proves the Theorem. □

The familiar example below shows that $\eta^R_1(M, N)$ need not vanish, even under the assumptions (4.1), if $\dim R$ is odd.

Example 4.6 [12, Example 1.5] The assumption that $c > 1$ is necessary in the Theorem. Let $R = \mathbb{C}[x, y, u, v]/(xu + yv)$ and set $M = R/(x, y)$, $N = R/(u, v)$, and $L = R/(x, v)$. Then $\eta^R_1(M, M) = \frac{1}{2}$, $\eta^R_1(M, N) = \frac{1}{2}$, and $\eta^R_1(M, L) = -\frac{1}{2}$.

Corollary 4.7 *Under the assumptions in (4.1) and for every pair of finitely generated, but not necessarily graded, R-modules M and N, if* $\dim R > 0$*, then $\eta^R_c(M, N)$ vanishes when $c > 1$. When* $\dim R = 0$*, then $\eta^R_c(M, N)$ vanishes for all c.*

Proof Upon passing to any faithfully flat field extension k' of k, the assumptions (4.1) remain valid, and, moreover, for finitely generated R-modules M and N, the value of $\eta^R_c(M, N)$ is unchanged. In more detail, since the lengths involved are dimensions over the field k for η^R_c and dimensions over the field k' for $\eta^{R \otimes_{\mathsf{k}} \mathsf{k}'}_c$, we have equality

$$\eta^R_c(M, N) = \eta^{R \otimes_{\mathsf{k}} \mathsf{k}'}_c(M \otimes_{\mathsf{k}} \mathsf{k}', N \otimes_{\mathsf{k}} \mathsf{k}').$$

In particular, by passing to the separable closure of k, we may assume that k is separably closed.

Since $\eta^R_c(-, -)$ is biadditive [7, Theorem 4.3] and defined for all pairs of finitely generated R-modules, it follows that η^R_c determines a bilinear pairing on $G(R)$, and hence on $G(R)_{\mathbb{Q}} := G(R) \otimes_{\mathbb{Z}} \mathbb{Q}$. It suffices to prove that this latter pairing is zero.

Assume $\dim R > 0$. From [14, Section 2.1], we recall the mapping from $K(X)_{\mathbb{Q}}$ to $G(R)_{\mathbb{Q}}$ given as follows: if T is a finitely generated graded R-module with associated coherent sheaf

$\widetilde{T}$ on X, then $K(X)_{\mathbb{Q}} \to G(R)_{\mathbb{Q}}$ sends $[\widetilde{T}] \in K(X)_{\mathbb{Q}}$ to $[T] \in G(R)_{\mathbb{Q}}$. As proven in [14, (2.4)], this mapping is onto, and hence the vector space $G(R)_{\mathbb{Q}}$ is spanned by classes of *graded* R-modules. Therefore, Theorem 4.5 applies to prove the pairing on $G(R)_{\mathbb{Q}}$ induced by η_c^R is the zero pairing.

Finally, if $\dim R = 0$, then $[R] = \dim_{\mathsf{k}}(R) \cdot [\mathsf{k}]$ in $G(R)_{\mathbb{Q}}$, and hence $[M] = \operatorname{length}(M) \cdot [\mathsf{k}]$ in $G(R)_{\mathbb{Q}}$. Since $\eta_c^R(R, R) = 0$ as R is projective, it follows that η_c^R vanishes for all pairs. $\square$

Corollary 4.8 *With the assumptions in (4.1), let M and N be finitely generated, but not necessarily graded, R-modules. Then for $c > 1$, the pair (M, N) is c-*Tor*-rigid; that is, if c consecutive torsion modules* $\operatorname{Tor}_i^R(M, N), \dots, \operatorname{Tor}_{i+c-1}^R(M, N)$ *all vanish for some $i \geq 0$, then* $\operatorname{Tor}_j^R(M, N) = 0$ *for $j \geq i$.*

Proof By Corollary 4.7, $\eta_c^R(M, N) = 0$ when $c > 1$, and the result immediately follows from [7, Theorem 6.3]. $\square$

Appendix A Adapting Gulliksen's Work

We show in this appendix how to modify Gulliksen's work in [10] to give an alternative proof of Proposition 2.1 from the body of this paper. The key result is Proposition A.1 below, which was originally proven by Dao [7, Lemma 3.2]. Using this result, a standard argument easily establishes Proposition 2.1.

Proposition A.1 *Assume Q is a noetherian ring, $f_1, \dots, f_c \in Q$ is a regular sequence, $R = Q/(f_1, \cdots, f_c)$ and M and N are finitely generated R-modules. If for all $i \gg 0$,* $\operatorname{Tor}_i^R(M, N)$ *has finite length and* $\operatorname{Tor}_i^Q(M, N) = 0$*, then there exists an integer j such that* $\bigoplus_{i \geq j} \operatorname{Tor}_i^R(M, N)$ *is artinian as a module over the polynomial ring $R[\chi_1, \dots, \chi_c]$, where the χ_i's act via the Eisenbud operators.*

Our proof of this result follows Gulliksen's proof of [10, Theorem 3.1]; we use two lemmas, both of which are analogues of his results. Our Lemma A.3 sidesteps the issue raised in [7, Example 2.9] while following [10, Lemma 1.2]. First we give some notation.

A.2 Let G be a $\mathbb{Z}$ graded ring concentrated in non-positive degrees (i.e., $G_i = 0$ for all $i > 0$). Note that given a graded G-module H, for any integer r, we have that $H_{<r} := \bigoplus_{i<r} H_i$ is a G-graded submodule of H and $H_{\geq r} := H/H_{<r}$ is a G-graded quotient module of H. Following Dao, we say that a graded G-module H is *almost artinian* if there is an integer r such that $H_{\geq r}$ is artinian as a G-module.

Lemma A.3 *Let G and H be as in paragraph A.2. Assume that H_i is an artinian G_0-module for all $i \gg 0$. Let $X\colon H \to H$ be a homogeneous G-linear map of negative degree. If* $\ker(X)$ *is almost artinian as a G-module, then H is almost artinian as a $G[X]$-module.*

Proof Let X have degree $w < 0$. For each r, there is a map of degree w on quotient modules given by multiplication by X:

$$X_{\geq r} : H_{\geq r} \to H_{\geq r+w}.$$

Note that $\ker(X_{\geq r}) = \ker(X)_{\geq r}$, where the latter uses the notation of A.2, and hence $\ker(X_{\geq r})$ is artinian as a G-module for $r \gg 0$, by assumption.

Since $w < 0$, there is a canonical surjection $\pi_{\geq r+w} \colon H_{\geq r+w} \twoheadrightarrow H_{\geq r}$ of graded G-modules having degree 0. Define $Y_{\geq r} = \pi_{\geq r+w} \circ X_{\geq r}$ so that $Y_{\geq r}$ is the endomorphism of $H_{\geq r}$ of degree w given by multiplication by X, and we have the left exact sequence

$$0 \to \ker(X_{\geq r}) \to \ker(Y_{\geq r}) \xrightarrow{\cdot X} \ker(\pi_{\geq r+w}). \tag{4.5}$$

The module $\ker(\pi_{\geq r+w})$ may be regarded as a G_0-module via restriction of scalars along the inclusion $G_0 \hookrightarrow G$. As a G_0-module, $\ker(\pi_{\geq r+w})$ is a finite direct sum of H_i for $i = r+w, \dots, r$. Hence for $r \gg 0$, it is an artinian G_0-module by our assumption. So for $r \gg 0$, it follows that $\ker(\pi_{\geq r+w})$ must also be artinian as a G-module. Thus for $r \gg 0$, the module $\ker(Y_{\geq r})$ is an artinian G-module, as follows from the exact sequence (4.5).

It follows from [10, Lemma 1.2] that $H_{\geq r}$ is artinian as a graded $G[Y_{\geq r}]$-module, for $r \gg 0$; that is, H is an almost artinian $G[X]$-module. □

The *Koszul algebra* K associated to a regular sequence $f_1, \dots, f_c$ of elements of a commutative ring Q is defined to be the following DG Q-algebra: The underlying graded Q-algebra is the exterior algebra $\bigwedge_Q^*(Q^c)$ on the free Q-module Q^c, indexed so that $\bigwedge_Q^j(Q^c)$ lies in homological degree j. Let $T_1, \dots, T_c$ be the standard basis of Q^c. The differential ∂ of K is uniquely determined by setting $\partial(T_i) = f_i$ and requiring that it satisfy the Leibniz rule: $\partial(ab) = \partial(a)b + (-1)^{\deg a} a\partial(b)$.

The Koszul algebra comes equipped with a ring map, called the *augmentation*, to its degree zero homology, namely $H_0(K) = Q/(f_1, \dots, f_c) =: R$. Since $f_1, \dots, f_c$ is Q-regular, the augmentation $K \twoheadrightarrow R$ is a quasi-isomorphism, so that K is a DG algebra resolution of R over Q that is free as a Q-module. Recall that a *DG module* over K is a graded K-module L equipped with a differential d_L of degree minus one so that $d_L(ax) = \partial(a)x + (-1)^{|a|} a d_L(x)$ holds for all homogeneous elements $a \in K$ and $x \in L$. See [3] for more background material on DG algebras and DG modules.

A.4 Let K be Koszul algebra over Q associated to a regular sequence $f_1, \dots, f_c$ in Q and let $I = \ker(K \twoheadrightarrow R)$ be the augmentation ideal. Let L be a DG module over K that is graded free as a module over the graded ring underlying K. Let N be a finitely generated R-module. Define Y to be $L \otimes_Q R = L/(f_1, \dots, f_c)L$. For any subset $S = \{i_1, \dots, i_s\}$ of $\{1, \dots, c\}$, define $I_S = (T_{i_1}, \dots, T_{i_s})$ and set $Y^S = Y/I_S Y$. In particular, $Y^\emptyset = Y$ and $Y^{\{1,\dots,c\}} = Y/IY = L/IL = L \otimes_K R$. For each S, the complex Y^S is a complex of Q-modules, and in fact of R-modules.

A.5 Gulliksen shows [10, p. 176–8] that for $i \in S$ there is an exact sequence of complexes of Q-modules

$$0 \to Y^S \xrightarrow{T_i} Y^{S\setminus\{i\}} \to Y^S \to 0$$

that is degree-wise split exact. It follows that

$$0 \to Y^S \otimes_Q N \to Y^{S\setminus\{i\}} \otimes_Q N \to Y^S \otimes_Q N \to 0 \tag{4.6}$$

is also exact, giving a long exact sequence in homology. The boundary map in this sequence, $\mathrm{H}(Y^S \otimes_Q N) \to \mathrm{H}(Y^S \otimes_Q N)$, is, up to sign, the action of X_i on $\mathrm{H}(Y^S \otimes_Q N)$ as defined by Gulliksen. Thus X_i has degree -2, since T_i has degree $+1$. Gulliksen proves that these actions commute: $X_i X_j = X_j X_i$ on $\mathrm{H}(Y^S \otimes_Q N)$ when $i, j \in S$. When $S = \{1, \dots, c\}$, these actions endow $\mathrm{H}(L/IL \otimes_R N)$ with the structure of a graded module over the graded ring $R[X_1, \dots, X_c]$ where each X_i has degree -2.

Our next lemma is similar to [10, Lemma 3.2(ii)].

Lemma A.6 *With the assumptions in paragraph A.4, and with the X_i from paragraph A.5, if $\mathrm{H}_i(L/IL \otimes_R N)$ is artinian as an R-module and $\mathrm{H}_i(L \otimes_Q N) = 0$ for $i \gg 0$, then $\mathrm{H}(L/IL \otimes_R N)$ is almost artinian as an $R[X_1, \ldots, X_c]$-module.*

Proof We have that for $i \gg 0$, the R-module $\mathrm{H}_i(Y^{\{1,\ldots,c\}} \otimes_R N) = \mathrm{H}_i(L/IL \otimes_R N)$ is artinian and $\mathrm{H}_i(Y^{\emptyset} \otimes_Q N) = \mathrm{H}_i(L/(f_1, \ldots, f_c)L \otimes_Q N) = 0$.

We first observe that, for every $S \subseteq \{1, \ldots, c\}$, $\mathrm{H}_i(Y^S \otimes_Q N)$ is also artinian over R, for $i \gg 0$. Indeed, this follows immediately by descending induction on the cardinality of S using the long exact sequence in homology associated to the exact sequence (4.6).

For $S = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, c\}$, let $G^S = R[X_{i_1}, \ldots, X_{i_s}]$. In particular, $G^{\emptyset} = R$ and $G^{\{1,\ldots,c\}} = R[X_1, \ldots, X_c]$. We prove $\mathrm{H}(Y^S \otimes_Q N)$ is an almost artinian G^S-module for each S, by induction on the cardinality of S. When $S = \emptyset$, then $Y^S = Y^{\emptyset} = L/(f_1, \ldots, f_c)L$ and, by assumption, $\mathrm{H}(Y^{\emptyset} \otimes_Q N) = \mathrm{H}(L/(f_1, \ldots, f_c)L \otimes_Q N)$ is an almost artinian R-module.

Assume that $i \in S$. The exact sequence (4.6) gives an exact homology sequence

$$\mathrm{H}(Y^{S\setminus\{i\}} \otimes_Q N) \longrightarrow \mathrm{H}(Y^S \otimes_Q N) \xrightarrow{X_i} \mathrm{H}(Y^S \otimes_Q N).$$

By the induction hypothesis, $\mathrm{H}(Y^{S\setminus\{i\}} \otimes_Q N)$ is almost artinian as a $G^{S\setminus\{i\}}$-module, and since the graded quotient of an almost artinian module is almost artinian, we see that $\ker(X_i)$ is almost artinian. Since $\mathrm{H}_q(Y^S \otimes_Q N)$ is artinian over $G_0 = R$, for $q \gg 0$, as was shown above, Lemma A.3 applies to show $\mathrm{H}(Y^S \otimes_Q N)$ is an almost artinian G^S-module. □

Proof of Proposition A.1 Regard M as a DG module concentrated in degree 0 via restriction of scalars along the augmentation. Gulliksen shows in [10, Lemma 2.4] how to construct a DG module L over K and a map $L \to M$ of DG modules such that L is free over the graded ring underlying K and the map $L \to M$ is a quasi-isomorphism. Note that $L \to M$ is, in particular, a resolution of M by free Q-modules. Moreover, Gulliksen shows [10, Lemma 2.6] that the projection map $L \twoheadrightarrow L/IL = L \otimes_K R$ is a quasi-isomorphism where $I = \ker(K \twoheadrightarrow R)$ is the augmentation ideal. In particular, this means that L/IL is an R-free resolution of M. We therefore obtain the isomorphisms

$$H_i(L \otimes_Q N) \cong \mathrm{Tor}_i^Q(M, N) \quad \text{and} \quad H_i(L/IL \otimes_R N) \cong \mathrm{Tor}_i^R(M, N)$$

for any R-module N.

Lemma A.6 now applies to prove that $\bigoplus_i \mathrm{Tor}_i^R(M, N)$ is almost artinian as a $R[X_1, \ldots, X_c]$-module. Finally, Avramov and Sun [4, §4] prove that the X_i's as constructed by Gulliksen agree with the Eisenbud operators, up to a sign. □

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