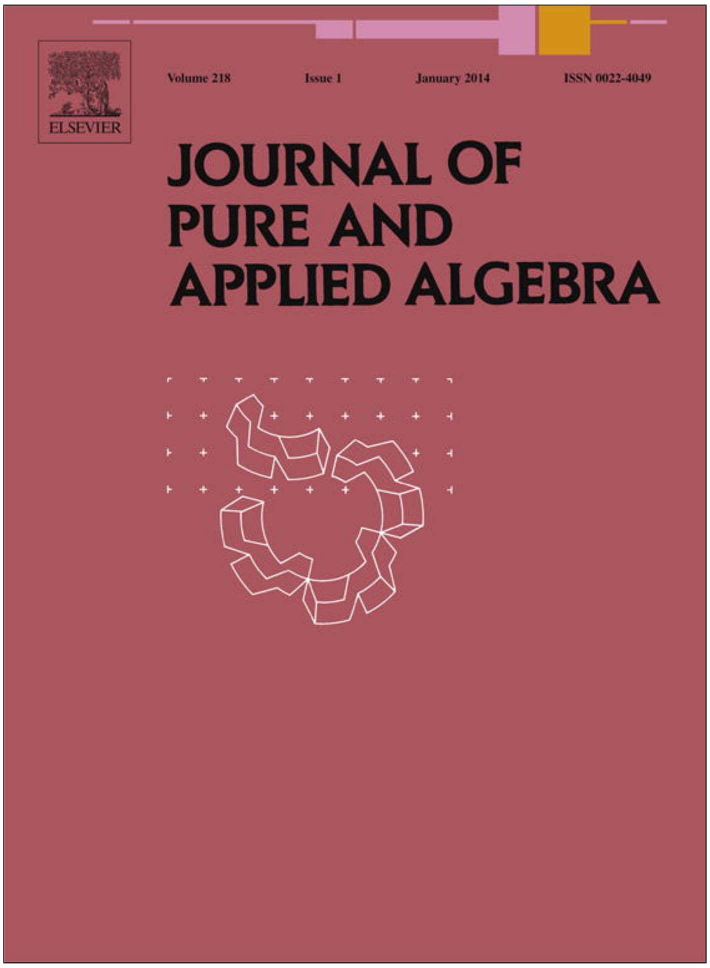
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Finiteness conditions on the Yoneda algebra of a monomial algebra

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ABSTRACT

Let A be a connected graded noncommutative monomial algebra. We associate to A a finite graph $\Gamma(A)$ called the CPS graph of A. Finiteness properties of the Yoneda algebra $\operatorname{Ext}_A(k, k)$ including Noetherianity, finite GK dimension, and finite generation are characterized in terms of $\Gamma(A)$. We show that these properties, notably finite generation, can be checked by means of a terminating algorithm.

1. Introduction

Complete intersections are a well-studied class of commutative algebras, yet there is not an agreed upon notion of complete intersection in the case of noncommutative algebras. From the point of view of noncommutative algebraic geometry, such a generalization should be homological. A starting point for a homological definition of complete intersection is the statement that for a graded Noetherian commutative k-algebra, the following properties are equivalent:

(i) A is a graded complete intersection
(ii) $\operatorname{Ext}_A(k, k)$ is a Noetherian k-algebra
(iii) $\operatorname{Ext}_A(k, k)$ has finite Gelfand–Kirillov (GK) dimension.

This equivalence derives from the combined work of several authors. To the best of our knowledge, (i)⇒(iii) is due to Tate [18], Gulliksen proved (i)⇒(ii) [14] and (iii)⇒(i) [12,13], and (ii)⇒(i) first appeared in Bøgvad–Halperin [4].

However, conditions (ii) and (iii) are not equivalent for graded Noetherian k-algebras, in fact, not even for algebras with monomial relations. Since such algebras are a tractable class of algebras with a well-understood projective resolution of the trivial module (see, for example [2,5]), their Yoneda algebras are computable, though often complex. This paper concerns the study of conditions (ii) and (iii) as well as the finite generation of $\operatorname{Ext}_A(k, k)$ when A is a connected graded noncommutative k-algebra with finitely many monomial relations.

To a monomial algebra A, we associate a finite directed graph $\Gamma(A)$ which we call the *CPS graph* of A. See Construction 2.1 for the definition of $\Gamma(A)$. Our first result concerns the Gelfand–Kirillov dimension of the Yoneda algebra $E(A) = \operatorname{Ext}_A(k, k)$.

Theorem 1.1 (*Corollary 2.8*). *Let A be a monomial k-algebra. If no pair of distinct circuits in $\Gamma(A)$ have a common vertex, then* $\operatorname{GKdim} E(A)$ *is the maximal number of distinct circuits contained in any walk. Otherwise,* $\operatorname{GKdim} E(A) = \infty$.

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Given any connected graded k-algebra B one can use a noncommutative Gröbner basis to associate to B a monomial algebra B' with the property GKdim $E(B) \le$ GKdim $E(B')$. Thus, Theorem 1.1 can provide an easily calculated upper bound on GKdim $E(B)$, though this bound is not always finite when GKdim $E(B)$ is. See Remark 2.9 below.

The Yoneda product on $E(A)$ can also be described combinatorially in terms of walks in the graph $\Gamma(A)$. Up to a notion of equivalence described in Section 2, all nonzero Yoneda products are compositions of admissible walks in $\Gamma(A)$. Using this description of the Yoneda product, we are able to characterize finite generation and the Noetherian property in $E(A)$. See Sections 2 and 3 for definitions of terminology and notation.

Theorem 1.2. *Let A be a monomial k-algebra.*

(1) *(Theorem 3.6) $E(A)$ is finitely generated if and only if for every infinite anchored walk p in $\Gamma(A)$, $\widetilde{p}$ contains a dense edge or two admissible edges of opposite parity.*
(2) *(Theorem 5.2) $E(A)$ is left (resp. right) Noetherian if and only if every vertex of $\Gamma(A)$ lying on an oriented circuit has out-degree (resp. in-degree) one and every edge of every oriented circuit is admissible.*

The second statement extends a theorem of Green et al. [10] who characterized Noetherianity of $E(A)$ in terms of the Ufnarovski relation graph of A in the case where A is quadratic.

Theorem 1.2(1) describes an infinite set of criteria to be satisfied for $E(A)$ to be finitely generated. Whether finite generation of $E(A)$ can be determined by finitely many criteria is a problem of recent interest. Working in the more general context of monomial factor algebras of quiver path algebras, Green and Zacharia [11] describe a (potentially infinite) process by which finite generation of the Yoneda algebra can be checked. Further progress was made by Davis [7] and Cone [6] who showed that finite generation can be determined by finitely many criteria when the given quiver is a cycle or an "in-spoked" cycle. In Section 4 we show the same can be said in our situation; that is, when the quiver consists of a single vertex and finitely many loops.

Theorem 1.3 (*Theorem 4.3*)**.** *Let A be a monomial k-algebra with* gl.dim $A = \infty$. *Let N be the smallest even integer greater than or equal to $2\mathcal{E}^2 + \mathcal{E} + 1$ where $\mathcal{E}$ is the number of edges in $\Gamma(A)$. The Yoneda algebra $E(A)$ is finitely generated if and only if every anchored walk of length N or $N + 1$ is decomposable.*

In our experience, determining if $E(A)$ is finitely generated when $E(A)$ has infinite Gelfand–Kirillov dimension can be a difficult problem, and we were unable to obtain an efficient bound in Theorem 1.3. However, the case GKdim $E(A) < \infty$ is much simpler. We describe a recursive algorithm for determining finite generation in that case in Section 4.

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2. The CPS graph

In [16], C. Phan associated a weighted digraph to any monomial graded algebra A. One important feature of Phan's graph is that a k-basis for $E(A)$ is represented by certain directed paths. After establishing some notation, we recall the unweighted version of Phan's graph – which we call the *CPS graph* of A – and we record a description of a minimal graded projective resolution of ${}_Ak$ (due to Cassidy and Shelton) in terms of this graph. We also prove several combinatorial facts about the CPS graph needed later.

Let k be a field. Throughout this paper we use the phrase *graded k-algebra* or just *k-algebra* to mean a connected, $\mathbb{N}$-graded, locally finite-dimensional k-algebra which is finitely generated in degree 1. If A is a graded k-algebra, we use the term *(left or right) ideal* to mean a graded (left or right) ideal of A generated by homogeneous elements of degree at least 2, unless otherwise indicated. The augmentation ideal is $A_+ = \bigoplus_{i\ge 1} A_i$. We abuse notation and use k (or ${}_Ak$ or k_A) to denote the trivial graded A-module A/A_+. The bigraded Yoneda algebra of A is the k-algebra $E(A) = \bigoplus_{i,j\ge 0} E^{i,j}(A) = \bigoplus_{i,j\ge 0} \mathrm{Ext}_A^{i,j}(k, k)$. (Here i denotes the cohomology degree and j denotes the internal degree inherited from the grading on A.) Let $E^p(A) = \bigoplus_q E^{p,q}(A)$.

Let $s \in \mathbb{N}$ and let $V = \mathrm{span}_k\{x_1, \dots, x_s\}$. We denote the tensor algebra on V by $T(V)$. The tensor algebra is a graded k-algebra, graded by tensor degree. We denote the tensor degree of a homogeneous element $w \in T(V)$ by $\deg w$. By a *monomial* in $T(V)$ we mean a pure tensor with coefficient 1. We consider $1_{T(V)}$ a monomial. By a *monomial algebra*, we mean an algebra of the form $A = T(V)/I$ where I is an ideal of $T(V)$ generated by finitely many monomials. Such an algebra A is a graded k-algebra with the grading inherited from the tensor grading on $T(V)$.

Let M be the set of monomials in $T(V)$. Multiplication in $T(V)$ induces the structure of a monoid on M. Let $I = \langle w_1, \dots w_r\rangle$ be an ideal in $T(V)$. We assume that the w_i form a minimal set of monomial generators for I and we let $d_i = \deg w_i$ be the tensor degree of w_i for each i. Recall that we assume every $d_i \ge 2$. Let $A = T(V)/I$ and let $\pi : T(V) \to A$ be the natural surjection.

Construction 2.1 (*CPS Graph*)**.** Suppose $m, w \in M - I$ and $w \otimes m \in I$. Let $L(w, m) = w'$ where $w = w'' \otimes w'$ for $w', w'' \in M$ and w' is minimal such that $w' \otimes m \in I$. For $m \in M - I$ define

$$\mathfrak{A}_m = \{w \in M - I : w \otimes m \in I \text{ and } L(w, m) = w\}.$$

Then the images of elements of $\mathfrak{A}_m$ in A generate the left annihilator of $\pi(m)$.

Let $\mathfrak{G}_0 = \{x_1, \dots x_s\}$ and for $i \geq 1$ let $\mathfrak{G}_i = \bigcup_{m \in \mathfrak{G}_{i-1}} \mathfrak{A}_m$. Finally, let $\mathfrak{G} = \bigcup_{i \geq 0} \mathfrak{G}_i$. Define the *CPS graph* of A to be the directed graph $\Gamma(A)$ with vertex set $\mathfrak{G}$, and edges $m_1 \to m_2$ whenever $m_2 \in \mathfrak{A}_{m_1}$.

We note that the graph $\Gamma(A)$ is finite. The graph may have loops and parallel edges with opposite orientation, but it has no parallel edges with the same orientation.

Example 2.2. Let $A = k\langle a, b, c, d\rangle / \langle abc, cdab\rangle$ and $B = A/\langle bcda\rangle$. For A we have $\mathfrak{G}_0 = \{a, b, c, d\}$, $\mathfrak{G}_1 = \{ab, cda\}$, $\mathfrak{G}_2 = \{ab, cd\}$, and $\mathfrak{G}_3 = \{ab, cd\} = \mathfrak{G}_i$ for $i > 3$. For the algebra B, $\mathfrak{G}_0 = \{a, b, c, d\}$, $\mathfrak{G}_1 = \{ab, cda, bcd\} = \mathfrak{G}_i$ for odd $i > 1$, and $\mathfrak{G}_2 = \{a, b, cd\} = \mathfrak{G}_j$ for even $j > 2$. The graphs $\Gamma(A)$ and $\Gamma(B)$ are shown below.

$\Gamma(A)$

$$\begin{tikzcd}
c \arrow[r] & ab \arrow[r, bend left] & cd \arrow[l, bend left] \\
b \arrow[r] & cda \arrow[u] & \\
a & & \\
d & &
\end{tikzcd}$$

$\Gamma(B)$

$$\begin{tikzcd}
c \arrow[r] & ab \arrow[r, bend left] & cd \arrow[l, bend left] \\
b \arrow[r, bend left] & cda \arrow[l, bend left] & \\
a \arrow[r, bend left] & bcd \arrow[l, bend left] & \\
d & &
\end{tikzcd}$$

Remark 2.3. An obvious, but extremely important feature of the CPS graph is that there is a directed edge $m_1 \to m_2$ with $m_1 \in \mathfrak{G}_0$ if and only if $m_2 \otimes m_1$ is a minimal generator of I. As illustrated by Proposition 2.7 below, this correspondence parallels the standard identification of $\mathrm{Ext}^2_A(k, k)$ with the graded dual of the space $I/(V \otimes I + I \otimes V)$.

If the defining relations of a monomial algebra A are quadratic, A is Koszul [17]. In that case, $\Gamma(A)$ is Ufnarovski's "relation graph" [19] for the Koszul dual algebra $A^!$. We also note that because we consider only minimal left annihilators, the CPS graph $\Gamma(A)$ is quite different from the notion of "zero-divisor graph" studied recently in [1].

We adopt some standard graph-theoretic terminology. By a *walk* we mean a finite or infinite sequence $v_0v_1v_2\cdots$ of vertices where $v_i \to v_{i+1}$ is a directed edge for all $0 \leq i < n$. If $v_0v_1\cdots v_n$ is a finite walk, we say the walk has *length n*. A walk is called an *edge path* or simply a *path* if it contains no repeated edges. By a *closed walk of length n* we mean a walk of length n such that $v_n = v_0$. A *circuit of length n* is a closed walk of length n such that $v_0, \dots, v_{n-1}$ are distinct. In the context of a weighted digraph, we abuse this terminology slightly and use "walk", "path", and "circuit" to refer to sequences of vertices in the underlying unweighted graph. If p and q are walks of length n and m respectively, we say *p extends q* or *q* is a *prefix* of p and write $q \vdash p$ if $n \geq m$ and $p_i = q_i$ for all $0 \leq i \leq m$.

In [5, Section 5], Cassidy and Shelton give a combinatorial description of a minimal graded projective left A-module resolution $P_\bullet$ of ${}_Ak$ in terms of monomial matrices. We briefly recount their resolution here, indexing the bases of each graded projective module by certain walks in $\Gamma(A)$.

For ease of exposition, we make the following definition.

Definition 2.4. A walk w in $\Gamma(A)$ is called *anchored* if $w_0 \in \mathfrak{G}_0$.

Remark 2.5. Anchored walks of length i in $\Gamma(A)$ correspond to the sets Γ_i described in [11].

Let $\mathcal{W}_n$ denote the set of all anchored walks of length n in $\Gamma(A)$. For each $w \in \mathcal{W}_n$, let $d_w = \sum_{i=0}^n \deg w_i$ where $\deg w_i$ denotes the tensor degree of the monomial w_i. Let $A(-d_w)$ be the graded free left A-module of rank 1 with grading shift $A(-d_w)_p = A_{p-d_w}$. Choose a basis for $A(-d_w)$ and denote this element by e_w. Let $P_0 = A$ be the graded free module with fixed basis element $e_\emptyset$ and for $j > 0$, let

$$P_j = \bigoplus_{w \in \mathcal{W}_{j-1}} A(-d_w).$$

Define $d_j : P_j \to P_{j-1}$ on the A-basis $\{e_w : w \in \mathcal{W}_{j-1}\}$ by setting $d_j(e_w) = \pi(w_{j-1})e_{\bar{w}}$ where $\bar{w} = w_0 \cdots w_{j-2}$ if $j \geq 2$ and $\bar{w} = \emptyset$ if $j = 1$. Extend d_j A-linearly to all of P_j. Since $w_{j-1} \to w_j$ is an edge in $\Gamma(A)$ only if $w_j \in \mathfrak{A}_{w_{j-1}}$, it is clear that $d_jd_{j+1} = 0$ for $j \geq 0$. The following lemma is a straightforward consequence of the definition of $\Gamma(A)$.

Lemma 2.6. *The complex $(P_\bullet, d_\bullet)$ described above is a minimal graded projective resolution of ${}_Ak$.*

Moreover, the bases for the P_j can be ordered so the matrices of the d_j with respect to the ordered bases are precisely the monomial matrices described in [5]. The next fact follows immediately from Lemma 2.6.

Proposition 2.7. *Let A be a monomial k-algebra and $i \in \mathbb{N}$. Then the graded duals $\{\varepsilon_w\}$ of the basis elements $\{e_w\}$ where w is an anchored walk of length i in $\Gamma(A)$ form a k-basis for $\mathrm{Ext}_A^{i+1}(k,k)$.*

We make extensive use of these bases throughout the paper. In Section 3, it becomes useful to define the symbols e_w and ε_w for non-anchored "admissible" walks (see Definition 2.11). Nonetheless, these newly-defined symbols still refer to elements of the bases of Proposition 2.7.

Several properties of A and $E(A)$ are immediate from Proposition 2.7. We denote the Gelfand–Kirillov dimension of a k-algebra A by GKdim A.

Corollary 2.8. (1) *If $\Gamma(A)$ contains no circuit, then* gl.dim A *is equal to the length of the longest path in $\Gamma(A)$. Otherwise,* gl.dim $A = \infty$.
(2) GKdim $E(A) = \infty$ *if and only if $\Gamma(A)$ contains distinct circuits with a common vertex.*
(3) *If no pair of distinct circuits in $\Gamma(A)$ have a vertex in common, then* GKdim $E(A)$ *is the maximal number of circuits contained in any walk (ignoring multiplicity).*
(4) *The Hilbert series of $E(A)$ is a rational function.*

Proof. (1) is clear. (2)–(4) are standard (see [19]). □

Remark 2.9. To any connected graded k-algebra $B = T(V)/J$ (we do not assume that J is generated by monomials) one can associate a monomial algebra in the usual way. Choose an ordered basis of V and induce a total ordering of the monoid M via degree–lexicographic order. Let $\mathcal{F}$ be a noncommutative Gröbner basis of J with respect to this ordering. Let $ht(\mathcal{F})$ be the set of highest terms of elements of $\mathcal{F}$ and let $B' = T(V)/\langle ht(\mathcal{F})\rangle$. Let

$$P_B(y,z) = \sum_{p,q} \dim \mathrm{Ext}_B^{p,q}(k,k) y^p z^q$$

denote the Poincare series of B. From the well-known coefficientwise inequality $P_B(y,z) \le P_{B'}(y,z)$ (see Lemma 3.4 of [2]) we can deduce GKdim $E(B) \le$ GKdim $E(B')$. Equality holds in the important case where the Gröbner basis for J consists of homogeneous polynomials of the same degree (see Corollary 4.6 of [15]). Thus Corollary 2.8 can sometimes provide an easily calculated upper bound on GKdim $E(B)$. For further examples, see Section 6.

It is also interesting to note that $E(A)$ has either exponential or polynomial growth — this is the case for commutative k-algebras (see [9] for the characteristic 0 graded case and [3] for the case of a local ring with residue field k). Observe GKdim $E(A) =$ GKdim $E(B) = 1$ for the algebras A and B from Example 2.2.

Given a minimal projective resolution of ${}_Ak$, one can compute the Yoneda product of classes ε_1 and ε_2 in $E(A)$ by lifting a representative of ε_2 through the resolution to the appropriate cohomology degree and composing with a representative of ε_1. For a monomial algebra A, we wish to describe the Yoneda product combinatorially in terms of walks in the graph $\Gamma(A)$. To do this, we introduce a notion of walk equivalence as a combinatorial analog of lifting a representative through a projective resolution.

We call two walks $p = p_0 \cdots p_n$ and $q = q_0 \cdots q_m$ in a CPS graph $\Gamma(A)$ *equivalent* if $m = n$ and

$$p_n \otimes p_{n-1} \otimes \cdots \otimes p_0 = q_m \otimes q_{m-1} \otimes \cdots \otimes q_0$$

as elements of M. If p and q are equivalent, we write $p \sim q$. It is clear that $\sim$ is an equivalence relation on walks in $\Gamma(A)$.

Lemma 2.10. *Let $\Gamma(A)$ be a CPS graph, and let p and q be equivalent walks of length $n > 0$ in $\Gamma(A)$. Then*

(1) *the prefix walks $p_0 \cdots p_{2k+1}$ and $q_0 \cdots q_{2k+1}$ are equivalent for all $0 \le k \le \lfloor \frac{n}{2} \rfloor$.*
(2) *if n is even, then $p_n = q_n$.*
(3) *we have $p_{2k+1} \otimes p_{2k} = q_{2k+1} \otimes q_{2k}$ for all $0 \le k \le \lfloor \frac{n}{2} \rfloor$.*
(4) *if $\deg(p_0) \ge \deg(q_0)$, then*

$$\deg(p_i) \ge \deg(q_i) \text{ if } 0 < i \le n \text{ is even}$$
$$\deg(q_i) \ge \deg(p_i) \text{ if } 0 < i \le n \text{ is odd}$$

(5) *the walk q is unique if it is anchored.*

Proof. To prove (1), we induct on k. Let $k = 0$. By switching the variables p and q if necessary, there is no loss of generality in assuming $\deg(p_0) \ge \deg(q_0)$. Since

$$p_n \otimes \cdots \otimes p_0 = q_n \otimes \cdots \otimes q_0$$

there exists a unique monomial $m \in M - I$ such that $m \otimes q_0 = p_0$. We have $q_1 \otimes q_0 \in I$, $m \otimes q_0 \notin I$, and

$$p_n \otimes \cdots \otimes p_1 \otimes m \otimes q_0 = q_n \otimes \cdots \otimes q_1 \otimes q_0$$

so there is a unique monomial $m' \in M - I$ such that $q_1 = m' \otimes m$. Now, $m' \otimes p_0 = m' \otimes m \otimes q_0 = q_1 \otimes q_0 \in I$ and $L(q_1, q_0) = q_1$, so $L(m', p_0) = m'$ and $m' \in \mathfrak{A}_{p_0}$. Thus

$$\begin{aligned} p_n \otimes \cdots \otimes p_1 \otimes p_0 &= q_n \otimes \cdots \otimes q_1 \otimes q_0 \\ &= q_n \otimes \cdots \otimes m' \otimes p_0 \end{aligned}$$

and $p_1, m' \in \mathfrak{A}_{p_0}$, so $p_1 = m'$. Hence

$$p_1 \otimes p_0 = m' \otimes p_0 = q_1 \otimes q_0$$

as desired. For the induction step, assume that

$$p_{2k+1} \otimes \cdots \otimes p_0 = q_{2k+1} \otimes \cdots \otimes q_0$$

and $\deg(p_{2k+2}) \geq \deg(q_{2k+2})$ and proceed as in the base case. This completes the proof of (1).

Statements (2) and (3) follow immediately from (1).

We consider statement (4). In light of (2) and (3), it suffices to prove $\deg(q_i) \geq \deg(p_i)$ for $0 < i \leq n$ odd. Since $q_1 \otimes q_0 = p_1 \otimes p_0$ and $\deg(p_0) \geq \deg(q_0)$, it is clear that $\deg(q_1) \geq \deg(p_1)$. Thus the result holds for $n \leq 2$. Assume that $n > 2$ and for $0 < i < n - 1$ odd $\deg(q_i) \geq \deg(p_i)$. Since $q_i \otimes q_{i-1} = p_i \otimes p_{i-1}$, there exists $m \in M$ such that $q_i = p_i \otimes m$. Suppose toward contradiction that $\deg(p_{i+1}) < \deg(q_{i+1})$. Since $q_{i+2} \otimes q_{i+1} = p_{i+2} \otimes p_{i+1}$, there exists $m' \in M$, $\deg(m') > 0$ such that $q_{i+1} = m' \otimes p_{i+1}$. Since $p_{i+1} \otimes p_i \in I$, we have $p_{i+1} \otimes q_i = p_{i+1} \otimes p_i \otimes m \in I$. The fact that $\deg(m') > 0$ contradicts the assumption that $L(q_{i+1}, q_i) = q_{i+1}$. So $\deg(q_{i+1}) \leq \deg(p_{i+1})$ and hence $\deg(q_{i+2}) \geq \deg(p_{i+2})$. Statement (4) now follows by induction.

To prove (5), suppose q' is another walk such that $p \sim q'$ and $q'_0 \in \mathfrak{G}_0$. Then $q \sim q'$ and $q_1 \otimes q_0 = q'_1 \otimes q'_0$. Since $\mathfrak{G}_0$ consists solely of degree 1 monomials, $q_0 = q'_0$ so $q_1 = q'_1$.

Suppose inductively that $q_i = q'_i$ for all $0 \leq i \leq 2k + 1 < n$. If $n = 2k + 2$, the induction hypothesis and the definition of equivalence imply that $q_{2k+2} = q'_{2k+2}$.

If $n > 2k + 2$, $q_{2k+3} \otimes q_{2k+2} = q'_{2k+3} \otimes q'_{2k+2}$. By switching the variables q and q' if necessary, we can assume $\deg(q_{2k+2}) \geq \deg(q'_{2k+2})$, so $q_{2k+2} = m \otimes q'_{2k+2}$ for some $m \in M$. But $L(q_{2k+2}, q_{2k+1}) = q_{2k+2}$ and

$$q'_{2k+2} \otimes q_{2k+1} = q'_{2k+2} \otimes q'_{2k+1} \in I$$

by the induction hypothesis. Thus $m = 1$, $q_{2k+2} = q'_{2k+2}$, and hence $q_{2k+3} = q'_{2k+3}$. Statement (5) now follows by induction. $\square$

Since the finite anchored walks in $\Gamma(A)$ enumerate a k-basis for $E(A)$, we make the following definition.

Definition 2.11. A finite walk in $\Gamma(A)$ is called *admissible* if it is equivalent to an anchored walk.

By Lemma 2.10(5), every admissible walk is equivalent to a unique anchored walk. In Example 2.2, the edge $ab \to cd$ is an admissible walk of length 1 in both $\Gamma(A)$ and $\Gamma(B)$, but the edge $cd \to ab$ is admissible in neither graph. In Phan's original weighted digraph, the edge weighting distinguished admissible edges from their counterparts. That distinction is too coarse for our purposes, but the importance of admissible edges seems evident from the following useful facts about admissible walks. Recall that if p and q are walks of length n and s respectively, we say q extends p if $n \leq s$ and $q_i = p_i$ for all $0 \leq i \leq n$.

Proposition 2.12. *Let $\Gamma(A)$ be a CPS graph, and let p be an admissible walk of length n in $\Gamma(A)$. Let q be a walk of length s such that q extends p. If either n or $s - n$ is even, then q is admissible.*

Proof. An admissible walk of length 0 consists of a single vertex in $\mathfrak{G}_0$, so the statement is trivial if $n = 0$. The statement is also trivial if $s = n$. So assume $n > 0$, $s - n > 0$, and let r be a path in $\Gamma(A)$ such that $p \sim r$ and $r_0 \in \mathfrak{G}_0$.

If n is even, then $r_n = p_n$ by Lemma 2.10(2). It follows immediately that the path $r' = r'_0 \cdots r'_s$ given by $r'_i = r_i$ for $0 \leq i \leq n$ and $r'_i = q_i$ for $n + 1 \leq i \leq s$ is equivalent to q and has $r'_0 \in \mathfrak{G}_0$.

Suppose n and s are odd. If $r_n = p_n$, we can proceed as above, so assume that $r_n \neq p_n$. By Lemma 2.10(3) and (4), there exists a monomial $m \in M$, $\deg(m) > 0$ such that $r_n = p_n \otimes m = q_n \otimes m$. Thus $q_{n+1} \otimes r_n \in I$. Put $r_{n+1} = L(q_{n+1}, r_n)$ and let $m' \in M$ such that $q_{n+1} = m' \otimes r_{n+1}$. Now,

$$q_{n+2} \otimes m' \otimes r_{n+1} = q_{n+2} \otimes q_{n+1} \in I.$$

Since $q_{n+1} \notin I$ and $L(q_{n+2}, q_{n+1}) = q_{n+2}$, it follows that $q_{n+2} \otimes m' \in \mathfrak{A}_{r_{n+1}}$. Put $r_{n+2} = q_{n+2} \otimes m'$. Then by construction, $r_0 \cdots r_{n+2}$ is a well-defined walk equivalent to $p' = q_0 \cdots q_{n+2}$ and $r_0 \in \mathfrak{G}_0$. Thus p' is an admissible walk of length $n + 2$ and q is an extension of p' of length s. The result now follows by induction on $s - n$. $\square$

We show in the next section that $E(A)$ is finitely generated if $\Gamma(A)$ has "enough" admissible walks. To make this more precise, we make the following definition.

Definition 2.13. Let p be an infinite walk in $\Gamma(A)$ and let $e = p_i p_{i+1}$ be an admissible edge in p. We call e *dense* in p if e has an admissible even-length extension in p.

In Example 2.2, $c \to ab \to cd \to ab \to cd \cdots$ is the only infinite anchored walk in $\Gamma(A)$. The admissible edge $ab \to cd$ is dense in this walk since

$$ab \to cd \to ab \sim b \to cda \to ab.$$

However, the edge $ab \to cd$ is not dense in the same walk in $\Gamma(B)$. The equivalent anchored walks corresponding to odd-length extensions of $ab \to cd$ begin at vertex b and end at either b or cda. It follows that no even length extension of $ab \to cd$ is admissible because condition (2) of Lemma 2.10 cannot be satisfied.

An admissible edge e may belong to many infinite walks. The edge e may be dense in some infinite walks, but not others. Furthermore, e may not be dense in an infinite walk w, but w may contain some other dense edge. See Example 6.1.

The following criterion for establishing density is immediate from Lemma 2.10(2).

Lemma 2.14. *Let w be a (possibly infinite) walk in $\Gamma(A)$ and let $e = w_i w_{i+1}$ be an admissible edge in w. Let q be any odd-length extension of e in w and let q' be the unique anchored walk equivalent to q. Then every even-length extension of q in w is admissible if and only if $q'_t = w_{i+t}$ for some even $t \geq 0$.*

3. Multiplicative structure

In this section we show certain extensions of walks in $\Gamma(A)$ correspond to Yoneda products in $E(A)$ and use the result to combinatorially characterize finite generation of $E(A)$. Recall from Proposition 2.7 that if w is an anchored walk of length n in $\Gamma(A)$, we denote the corresponding A-basis element of P_{n+1} by e_w and the graded dual of e_w by ε_w.

Fix an anchored walk q of length n. To connect the Yoneda product in $E(A)$ to extensions of walks in $\Gamma(A)$, we explicitly construct lifts of ε_q through the resolution $(P_\bullet, d_\bullet)$ defined in Section 2. We need one additional definition before describing the construction.

Definition 3.1 ([5]). An element r in an ideal $I \subset T(V)$ is called *essential* if r is not in the ideal generated by $V \otimes I + I \otimes V$.

We note that a monomial r in a monomial ideal I is essential if and only if r is a minimal generator of I. Hence a walk $w_0 w_1$ in $\Gamma(A)$ is admissible if and only if $w_1 \otimes w_0$ is a minimal generator of I.

For $i \geq 0$, we define A-module maps f_i such that the following diagram commutes.

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+3} \arrow[r, "d_{n+3}"] \arrow[d, "f_2"'] & P_{n+2} \arrow[r, "d_{n+2}"] \arrow[d, "f_1"'] & P_{n+1} \arrow[d, "f_0"'] \arrow[dr, "\varepsilon_q"] & \\ \cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r] & k \end{tikzcd} \tag{1}$$

For $i \geq 0$ let Q_{n+i} be the graded free submodule of P_{n+i} spanned by the set $\{e_r : q \vdash r\}$ and let Z_{n+i} be the complement to Q_{n+i} in P_{n+i}. We observe that $P_{\geq n} = Q_{\geq n} \oplus Z_{\geq n}$ as complexes of graded free left A-modules. For all $i \geq 0$, we define $f_i(Z_{n+i+1}) = 0$.

We define f_i on the specified A-basis of Q_{n+i+1} in several steps.

(i) Define $f_0(e_q) = e_\emptyset$.

(ii) For any walk r such that $e_r \in Q_{n+2}$, we have $r_{n+1} = m \otimes x_j$ for a unique $m \in M$ and generator x_j. Define $f_1(e_r) = \pi(m)e_{x_j}$.

(iii) Suppose $d > 2$ and r is a walk such that $e_r \in Q_{n+d}$. Recall that the walk $r_{n+1}r_{n+2}$ is admissible if and only if $r_{n+2} \otimes r_{n+1}$ is essential. (See Remark 2.3.) If $r_{n+2} \otimes r_{n+1}$ is not essential, we define $f_{d-1}(e_r) = 0$. If $r_{n+2} \otimes r_{n+1}$ is essential, our construction depends on the parity of d.

If d is odd and $r_{n+1}r_{n+2}$ is admissible, the walk $r_{n+1} \cdots r_{n+d-1}$ of length $d-2$ is admissible by Proposition 2.12. Let $r' = r'_0 \cdots r'_{d-2}$ be the unique anchored walk equivalent to $r_{n+1} \cdots r_{n+d-1}$. Then $e_{r'} \in P_{d-1}$ and we define $f_{d-1}(e_r) = e_{r'}$.

If d is even and $r_{n+1}r_{n+2}$ is admissible, then the length $d-3$ walk $r_{n+1} \cdots r_{n+d-2}$ is admissible. Let $r'_0 \cdots r'_{d-3}$ be the equivalent anchored walk. Lemma 2.10(3) and (4) imply that there exists a unique monomial $m \in M$ such that $r_{n+d-2} \otimes m = r'_{d-3}$. Since $r_{n+d-1} \otimes r_{n+d-2} \in I$, we have $r_{n+d-1} \otimes r'_{d-3} \in I$. Put $r'_{d-2} = L(r_{n+d-1}, r'_{d-3})$ and let $m' \in M$ be the unique monomial such that $r_{n+d-1} = m' \otimes r'_{d-2}$. Then $r' = r'_0 \cdots r'_{d-2}$ is a well-defined anchored walk, $e_{r'} \in P_{d-1}$, and we may define $f_{d-1}(e_r) = \pi(m')e_{r'}$.

To summarize, for $d > 0$ and r a walk such that $e_r \in Q_{n+d}$, we define

$$f_{d-1}(e_r) = \begin{cases} e_\emptyset & \text{if } d = 1 \\ \pi(m)e_{x_j} & \text{if } d = 2 \text{ and } r_{n+1} = m \otimes x_j \\ e_{r'} & \text{if } r_{n+2} \otimes r_{n+1} \text{ is essential and } d > 2 \text{ odd} \\ \pi(m')e_{r'} & \text{if } r_{n+2} \otimes r_{n+1} \text{ is essential and } d > 2 \text{ even} \\ 0 & \text{else} \end{cases}$$

where r' is a uniquely determined anchored walk of length $d-2$ and $r_{n+d-1} = m' \otimes r'_{d-2}$. Extending the definitions of the f_i A-linearly, we obtain a sequence of A-module maps.

Lemma 3.2. *With f_i defined as above, the diagram (1) commutes.*

Proof. Because $f_i(Z_{n+i+1}) = 0$ for all $i \geq 0$ and $Z_{\geq n}$ is a subcomplex of $P_{\geq n}$, it suffices to show commutativity for the complex $Q_{\geq n}$. We compute the first few squares explicitly.

$(d=1)$ The augmentation map $\epsilon : P_0 \to k$ takes $e_\emptyset \mapsto 1$, so $\varepsilon_q = \epsilon f_0$.

$(d=2)$ Let r be a walk of length $n+1$ in $\Gamma(A)$ which extends q. Then

$$f_0 d_{n+2}(e_r) = f_0(\pi(r_{n+1})e_q) = \pi(r_{n+1})e_\emptyset.$$

On the other hand, $r_{n+1} = m \otimes x_j$ for unique $m \in M$ and generator x_j so

$$d_1 f_1(e_r) = d_1(\pi(m)e_{x_j}) = \pi(m)\pi(x_j)e_\emptyset = \pi(r_{n+1})e_\emptyset.$$

$(d=3)$ Let r be a walk of length $n+2$ in $\Gamma(A)$ which extends q. If $r_{n+2} \otimes r_{n+1}$ is essential, then

$$d_2 f_2(e_r) = d_2(e_{r'}) = \pi(r'_1)e_{\overline{r'}}$$

where $r' = r'_0 r'_1$ is anchored, equivalent to $r_{n+1}r_{n+2}$, and $\overline{r'} = r'_0$. On the other hand,

$$f_1(d_{n+3}(e_r)) = f_1(\pi(r_{n+2})e_{\bar{r}}) = \pi(r_{n+2})\pi(m)e_{x_j} = \pi(r_{n+2} \otimes m)e_{x_j}$$

where $\bar{r} = r_0 \cdots r_{n+1}$ and $r_{n+1} = m \otimes x_j$. In this case, since $r'_0 \in \mathfrak{G}_0$ and $r'_1 \otimes r'_0 = r_{n+2} \otimes r_{n+1}$, we have $r'_0 = x_j$ and $r'_1 = r_{n+2} \otimes m$ as desired.

If $r_{n+2} \otimes r_{n+1}$ is not essential, $f_2(e_r) = 0$ and $\pi(r_{n+2} \otimes m) = 0$ since $r_{n+2} \in \mathfrak{A}_{r_{n+1}}$.

For $d > 3$, the arguments are similar to those above and are omitted. The key observation is that $r'_0 \cdots r'_{d-4}$ is equivalent to $r_{n+1} \cdots r_{n+d-3}$ by Lemma 2.10(1) so $\bar{r}' = r'_0 \cdots r'_{d-4}$ by uniqueness (Lemma 2.10(5)). The definitions of the f_i then imply $r_{n+d-2} = m' \otimes r'_{d-3}$ if d is odd and $r_{n+d-1} = m' \otimes r'_{d-2}$ if d is even, from which commutativity follows. $\square$

We are ready to give a combinatorial description of the Yoneda composition product. Though our basis for $E(A)$ is enumerated by *anchored* walks, the most natural combinatorial interpretation of Yoneda composition in our setting dictates that we append a (not necessarily anchored) *admissible* walk to the end of an anchored walk. Consequently, we find it conceptually helpful and notationally convenient to define the symbol ε_w for any admissible walk w to mean the dual basis element ε_q where q is the unique anchored walk equivalent to w guaranteed by Lemma 2.10. If $\alpha, \beta \in E(A)$, we denote the Yoneda product by $\alpha \star \beta$.

Proposition 3.3. *Let $p = p_0 \cdots p_s$ and $q = q_0 \cdots q_n$ be admissible walks in $\Gamma(A)$. Then $\varepsilon_p \star \varepsilon_q = 0$ unless there exist walks $p' \sim p$ and $q' \sim q$ such that q' is anchored and $q'_n \to p'_0$ is an edge in $\Gamma(A)$. In that case $\varepsilon_p \star \varepsilon_q = \varepsilon_w$ where $w \sim q'_0 \cdots q'_n p'_0 \cdots p'_s$.*

Proof. By Lemma 2.10(5) it suffices to consider the case where q is anchored. For $i \geq 0$, let f_i be defined as above. By definition of Yoneda composition product, $\varepsilon_p \star \varepsilon_q = \varepsilon_p f_{s+1}$. Let r be any anchored walk of length $n+s+1$. If r does not extend q, then $\varepsilon_p f_{s+1}(e_r) = 0$. If r extends q, then

$$\varepsilon_p f_{s+1}(e_r) = \begin{cases} \varepsilon_p(\pi(m)e_{x_j}) & \text{if } s = 0 \text{ and } r_{n+1} = m \otimes x_j \\ \varepsilon_p(e_{r'}) & \text{if } s > 0 \text{ is odd and } r_{n+2} \otimes r_{n+1} \text{ is essential} \\ \varepsilon_p(\pi(m')e_{r'}) & \text{if } s > 0 \text{ is even and } r_{n+2} \otimes r_{n+1} \text{ is essential} \\ 0 & \text{else} \end{cases}$$

where r' and m' are defined as in (iii) above. Thus $\varepsilon_p f_{s+1}(e_r) = 0$ unless r extends q, $p \sim r'$, and, if s is even, $r'_s = r_{n+s+1}$. The last condition implies $r' \sim r_{n+1} \cdots r_{n+s+1}$ when s is even. (This equivalence always holds when s is odd.) So if $\varepsilon_p f_{s+1}(e_r) \neq 0$, we have $\varepsilon_p f_{s+1}(e_r) = 1$ and it follows that $\varepsilon_p \star \varepsilon_q = \varepsilon_w$ where $w = q_0 \cdots q_n r_{n+1} \cdots r_{n+s+1}$. Since $p \sim r' \sim r_{n+1} \cdots r_{n+s+1}$, setting $p' = r_{n+1} \cdots r_{n+s+1}$ gives the desired result. $\square$

We call a class $\alpha \in E^i(A)$ for $i > 0$ *decomposable* if α is in the subalgebra of $E(A)$ generated by $\bigoplus_{j<i} E^j(A)$. Otherwise we call α *indecomposable*. Proposition 3.3 illustrates a nice feature of our chosen k-basis for $E(A)$.

Corollary 3.4. *If w is an anchored walk of length n in $\Gamma(A)$, then ε_w is decomposable if and only if there exists an admissible walk $p = p_0 \cdots p_m$ such that $w = w_0 \cdots w_i p_0 \cdots p_m$ for some $0 \leq i < n$.*

We note this implies that the relations of $E(A)$ consist exclusively of monomials and binomials. That $E(A)$ can be presented this way appears as Theorem B in [11].

We are nearly ready to give a combinatorial characterization of finite generation. We call an admissible walk w *decomposable* (resp. *indecomposable*) if ε_w is decomposable (resp. indecomposable) in $E(A)$.

The following important fact is an application of the classical König's Lemma (see [8] p. 1046); its proof by induction is omitted.

Lemma 3.5. *If $\Gamma(A)$ contains infinitely many indecomposable finite anchored walks, there exists an infinite anchored walk with infinitely many indecomposable finite prefixes.*

Our main theorem characterizes infinite walks with infinitely many indecomposable prefixes. In the next section, we give a finite procedure for checking these conditions. If p is a walk of length n in $\Gamma(A)$, let $\widetilde{p} = p_1 \cdots p_n$ be the walk p with the initial edge deleted. Recall from Section 2 that if e is an admissible edge in an infinite walk p, we call e *dense* in p if e has an admissible even-length extension in p.

Theorem 3.6. *Let A be a monomial k-algebra. The following are equivalent.*

(1) *$E(A)$ is finitely generated.*
(2) *Every infinite anchored walk in $\Gamma(A)$ has finitely many indecomposable prefixes.*
(3) *For every infinite anchored walk p in $\Gamma(A)$, $\widetilde{p}$ contains a dense edge or two admissible edges of opposite parity.*

Here "opposite parity" means the number of edges properly between the two admissible edges is even. See Section 6 for an illustration of the theorem.

Proof. The equivalence of (1) and (2) follows from Lemma 3.5. We prove that (2) and (3) are equivalent.

Let p be an infinite anchored walk in $\Gamma(A)$. If $\widetilde{p}$ contains a dense edge e, then there exists an even-length extension $e \vdash q$ in p such that q is admissible. By Proposition 2.12, every extension of q is admissible, so by Corollary 3.4, p has only finitely many indecomposable prefixes.

By Proposition 2.12, every odd-length extension of an admissible edge is admissible. Hence if $\widetilde{p}$ has admissible edges of opposite parity, p has only finitely many indecomposable prefixes.

Suppose instead that $\widetilde{p}$ has no dense edges and all admissible edges in $\widetilde{p}$ have the same parity. If $\widetilde{p}$ contains no admissible edges, then Corollary 3.4 implies that every finite prefix of p is indecomposable. If $\widetilde{p}$ contains an admissible edge, let $e = p_ip_{i+1}$ be the admissible edge with i minimal. Since admissible edges have the same parity, for $n > 0$ the admissible edges in $p_0 \cdots p_{i+2n}$ have the form $p_{i+2j}p_{i+2j+1}$ for $0 \le j < n$. Since a walk of the form $p_{i+2j} \cdots p_{i+2n}$ has even length and $\widetilde{p}$ contains no dense edges, $p_0 \cdots p_{i+2n}$ is indecomposable for all $n > 0$ by Corollary 3.4. $\square$

Remark 3.7. If w is an infinite walk and for $j > 0$, w_jw_{j+1} is a dense edge in $\widetilde{w}$, then any admissible edge $w_{j-2i}w_{j-2i+1}$, $0 \le i \le \lfloor \frac{j}{2} \rfloor$ is also dense in w. This follows from the fact that $w_{j-2i} \cdots w_{j-1}$ is an odd-length extension of $w_{j-2i}w_{j-2i+1}$, Propositions 2.12, 2.7 and 3.3. Thus $\widetilde{w}$ contains a dense edge if and only if the first admissible edge in $\widetilde{w}$ is dense in w. It follows from the discussion in Section 2 that for the algebras A and B from Example 2.2, $E(A)$ is finitely generated and $E(B)$ is not.

4. An upper bound for checking finite generation

At first glance, verification of the conditions of Theorem 3.6 appears to require an infinite procedure, in general. The infinitude arises both from the number of infinite walks in $\Gamma(A)$ and the determination of edge density. In this section we establish an upper bound on the cohomological degree of an indecomposable element if $E(A)$ is finitely generated.

For the first time, the distinction between "path" and "walk" is important. Let L be the maximal length of an anchored edge path p in $\Gamma(A)$ with $p_{L-1}p_L$ an admissible edge, and no edge p_ip_{i+1} admissible for $0 < i < L-1$. Let S be the size of the largest edge equivalence class, and let $\mathcal{E}$ be the number of edges of $\Gamma(A)$. We note that $S = 1$ if and only if every admissible edge is anchored. The bound we obtain below depends on L, S, and $\mathcal{E}$. The bound stated in the Introduction is then obtained from the obvious inequalities $L \le \mathcal{E}$ and $S - 1 \le \mathcal{E}$.

Finite generation is easy to detect when $S = 1$, so we handle that case first.

Theorem 4.1. *Let A be a monomial k-algebra with* gl.dim $A = \infty$ *and $S = 1$. Then $E(A)$ is finitely generated if and only if every circuit in $\Gamma(A)$ contains a vertex in $\mathfrak{G}_0$.*

Proof. If every circuit in $\Gamma(A)$ contains a vertex in $\mathfrak{G}_0$, then for any infinite anchored walk w, the walk $\widetilde{w}$ contains a vertex in $\mathfrak{G}_0$. Thus $\widetilde{w}$ contains a dense edge, and because w was arbitrary, $E(A)$ is finitely generated by Theorem 3.6.

Since gl.dim $A = \infty$, the graph $\Gamma(A)$ contains a circuit. If $\Gamma(A)$ contains a circuit C missing $\mathfrak{G}_0$, let p be an anchored path of length $n > 0$ such that p_n is in C. Since $S = 1$, neither $\widetilde{p}$ nor C contains an admissible edge. Let q be the infinite extension of p defined by repeatedly traversing C. Then $\widetilde{q}$ contains no admissible edge; hence every prefix of q is indecomposable by Corollary 3.4 and $E(A)$ is not finitely generated by Theorem 3.6. $\square$

For the rest of this section, we assume $S > 1$, so non-anchored admissible edges exist and $L > 1$. Next we establish an important upper bound.

Lemma 4.2. *Suppose q is a walk of length $N > 2\mathcal{E}(S-1) + L$ in $\Gamma(A)$. Assume that $S > 1$ and $\widetilde{q}$ contains an admissible edge q_jq_{j+1} for $0 < j < L$. Let $p = q_jq_{j+1} \cdots q_N$ if $N - j$ is even and $p = q_jq_{j+1} \cdots q_{N-1}$ if $N - j$ is odd. Then*

(1) *p is admissible*
(2) *for all $0 \le i \le 2\mathcal{E}(S-1)$, we have $q_{2i+j}q_{2i+j+1} \sim p'_{2i}p'_{2i+1}$ where $p' \sim p$ and p' is anchored.*
(3) *Either*
   (a) *$q_{2i+j}q_{2i+j+1} = p'_{2i}p'_{2i+1}$ for some $0 \le i \le 2\mathcal{E}(S-1)$ or*
   (b) *there exist $0 \le c < d \le 2\mathcal{E}(S-1)$ such that*

$$q_{2c+j}q_{2c+j+1} = q_{2d+j}q_{2d+j+1} \quad \text{and} \quad p'_{2c}p'_{2c+1} = p'_{2d}p'_{2d+1}.$$

Proof. Statement (1) is immediate from Proposition 2.12 since the length of p is odd. Statement (2) then follows from Lemma 2.10. To prove (3), let $\mathbb{E}$ denote the set of edges of $\Gamma(A)$ and let

$$\mathcal{O} = \{(e_1, e_2) \in \mathbb{E} \times \mathbb{E} : e_1 \sim e_2, e_1 \neq e_2\}.$$

Then $|\mathcal{O}| \leq \mathcal{E}(S-1)$. Since the walks p and p' consist of at least $2\mathcal{E}(S-1)$ edges of $\Gamma(A)$, either one of the pairs of equivalent edges

$$(q_{2i+j}q_{2i+j+1}, p'_{2i}p'_{2i+1}) \qquad 0 \leq i \leq \mathcal{E}(S-1)$$

is not in $\mathcal{O}$, in which case (a) holds, or some element of $\mathcal{O}$ appears twice, in which case (b) holds. □

Theorem 4.3. *Let A be a monomial k-algebra with* gl.dim $A = \infty$ *and $S > 1$. Let N be the smallest even integer greater than or equal to $2\mathcal{E}(S-1) + L + 1$. The Yoneda algebra $E(A)$ is finitely generated if and only if every anchored walk q of length N or $N+1$ is decomposable.*

Since $S-1$ and L are at most $\mathcal{E}$, we obtain the weaker, but more easily stated bound of $2\mathcal{E}^2 + \mathcal{E} + 1$ mentioned in the Introduction.

Proof. Suppose every anchored walk of length N or $N+1$ is decomposable. Let q be any anchored walk of length $N+1$. Then q and $q' = q_0 \cdots q_N$ are both decomposable. By Corollary 3.4, $\widetilde{q'}$ contains an admissible edge q_iq_{i+1}. By Proposition 2.12, every odd length extension of q_iq_{i+1} is admissible. This can account for the decomposability of only one of q and q'. Since both are decomposable, either q contains an admissible edge whose parity is opposite q_iq_{i+1} or an even length extension of q_iq_{i+1} is admissible, making q_iq_{i+1} dense in any infinite walk with prefix q. Since q was arbitrary, $E(A)$ is finitely generated by Theorem 3.6.

Conversely, suppose $\Gamma(A)$ contains an indecomposable anchored walk q of length N or $N+1$. We will construct an infinite anchored walk w in $\Gamma(A)$ in which all admissible edges have the same parity, but none are dense in w. That $E(A)$ is not finitely generated will then follow from Theorem 3.6.

By the discussion preceding Lemma 4.1, we have $S > 1$; hence $\widetilde{q}$ contains a circuit. If $\widetilde{q}$ contains no admissible edge, or if the first admissible edge of $\widetilde{q}$ follows a circuit in $\widetilde{q}$, we can construct an infinite walk in $\Gamma(A)$ in which every prefix is indecomposable as in the proof of Lemma 4.1. Otherwise, let $0 < j < L$ be minimal such that q_jq_{j+1} is an admissible edge of $\widetilde{q}$.

Since q is indecomposable, Proposition 2.12 implies the length of q and the index j must have opposite parity. We consider only the case where q has length N, the case of length $N+1$ being identical after the obvious necessary index shift. Let $p = q_j \cdots q_{N-1}$. By Lemma 4.2(1), p is admissible, so let p' be the unique anchored walk equivalent to p.

The walk $q_j \cdots q_N$ is not admissible, so by Lemma 2.14 we must have $q_{2i+j} \neq p'_{2i}$ for all $0 \leq i \leq 2\mathcal{E}(S-1)$. Therefore, we have $q_{2i+j}q_{2i+j+1} \neq p'_{2i}p'_{2i+1}$ for all $0 \leq i \leq 2\mathcal{E}(S-1)$. By Lemma 4.2(3), there exist $0 \leq c < d \leq 2\mathcal{E}(S-1)$ such that $q_{2c+j}q_{2c+j+1} = q_{2d+j}q_{2d+j+1}$ and $p'_{2c}p'_{2c+1} = p'_{2d}p'_{2d+1}$.

Let $z = q_{2c+j} \cdots q_{2d+j-1}$, let $z' = p'_{2c} \cdots p'_{2d-1}$ and let w be the infinite walk

$$q_0 \cdots q_{2c+j-1}zzz \cdots.$$

Since $q_{2c+j} = q_{2d+j}$ and $q_{2d+j-1} \to q_{2d+j}$ is an edge in $\Gamma(A)$, the walk w is indeed well-defined. Likewise, the walk

$$w' = p'_0 \cdots p'_{2d-1}z'z'z' \cdots,$$

is well-defined. Since all admissible edges of $\widetilde{q}$ have the same parity as q_jq_{j+1}, the same is true for $\widetilde{w}$. Moreover, every admissible extension of q_jq_{j+1} in w is a prefix of w'. Since $q_{2i+j} \neq p'_{2i}$ for all $0 \leq i \leq d$ as noted above, the edge q_jq_{j+1} is not dense in w by Lemma 2.14. By Remark 3.7, $\widetilde{w}$ contains no dense edges. Therefore, $E(A)$ is not finitely generated by Theorem 3.6. □

In many cases, one can determine if $E(A)$ is finitely generated well before the upper bound above. Indeed if GKdim $E(A) = 1$, there are finitely many infinite anchored walks. If $d =$ GKdim $E(A) < \infty$, it is easy to describe a recursive procedure:

(1) Analyze the (finite number of) subgraphs of $\Gamma(A)$ with at most $d-1$ distinct circuits (ignoring multiplicity) in any walk.
(2) If no anchored walk with infinitely many indecomposable prefixes is found, let $\mathcal{M}$ be the maximal multiplicity of a circuit in an indecomposable walk. Analyze the (finite number of) infinite walks w containing d distinct circuits (ignoring multiplicity) such that the first $d-1$ circuits of w occur with multiplicity $\leq \mathcal{M}$.

5. The Noetherian property

Green et al. [10] observed that if A is a monomial quadratic algebra, it is possible to determine if $E(A)$ is Noetherian by considering its Ufnarovski relation graph. As noted in Section 2, if A is a monomial quadratic algebra, then $\Gamma(A)$ is precisely the Ufnarovski graph of the Koszul dual $A^! \cong E(A)$ with edge orientations reversed. In this section we prove an analog of Green et al.'s "Noetherianity" theorem (Theorem 5.4 of [10]) holds for $\Gamma(A)$.

The following lemma illustrates an important difference between quadratic monomial algebras and monomial algebras with defining relations in higher degrees. The lemma also conveys the sense in which Theorem 5.2 below generalizes the result in [10].

Lemma 5.1. *Let A be a monomial k-algebra such that the defining ideal of A is generated by quadratic monomials. Then $\mathfrak{G} = \mathfrak{G}_0$ and every edge of $\Gamma(A)$ is admissible.*

Proof. Since the minimal generators of I are quadratic, for any generator x_j, $\mathfrak{A}_{x_j}$ consists of linear monomials. Thus $\mathfrak{G}_1 \subset \mathfrak{G}_0$ and $\mathfrak{G} = \mathfrak{G}_0$. It follows (see Remark 2.3) that every edge of $\Gamma(A)$ is admissible. $\square$

For our discussion of the Noetherian property, we discard the assumption that ideals in a graded k-algebra are generated by homogeneous elements of degrees ≥ 2.

To establish the main theorem of this section in the left Noetherian case, we filter a left ideal by defining a total order on the path basis of Proposition 2.7. We invoke this total order only when $\Gamma(A)$ has the property that every vertex lying on an oriented circuit has out-degree 1. To handle the right Noetherian case, one first defines the analogous total order under the assumption that every vertex on an oriented circuit has in-degree 1. In the interest of brevity, we provide details only for the left Noetherian case. We define the order in several steps.

We first fix a total ordering of the s circuits of $\Gamma(A)$: $C_1 < C_2 < \cdots < C_s$. An *in-path* p for a circuit C_i is an anchored path p with the final vertex of p on C_i and no other vertex of p on C_i. The set of in-paths to a particular circuit is finite, so we fix a total ordering on each set of in-paths. Of the maximal paths in $\Gamma(A)$, finitely many terminate on no circuit. We fix a total ordering on these paths as well and define them to be less than any in-path.

If p and q are in-paths of lengths n and m for circuits C_i and C_j respectively, we define $p < q$ if $n < m$ or $n = m$ and $i < j$ or $n = m$ and $i = j$ and $p < q$ in the fixed ordering on in-paths of C_i.

If w is any anchored walk in $\Gamma(A)$, then there exists a unique path $\overline{w}$ such that exactly one of the following holds:

- $\overline{w}$ is an in-path terminating on C_i with i minimal and $\overline{w}$ extends w or
- $\overline{w}$ is an in-path terminating on C_i and is a proper prefix of w or
- $\overline{w}$ is a maximal extension of w terminating on no circuit.

If p and q are anchored walks of lengths n and m respectively, we define $\varepsilon_p < \varepsilon_q$ if $n < m$ or if $n = m$ and $\overline{p} < \overline{q}$.

Theorem 5.2. *For a monomial k-algebra A, the Yoneda algebra $E(A)$ is left (resp. right) Noetherian if and only if*

(1) *every vertex of $\Gamma(A)$ lying on an oriented circuit has out-degree (resp. in-degree) 1, and*
(2) *every edge of every oriented circuit is admissible.*

Proof. If $\Gamma(A)$ contains no circuit, $E(A)$ is finite dimensional, hence Noetherian, by Corollary 2.8. Assume $\Gamma(A)$ contains a circuit.

First suppose $\Gamma(A)$ satisfies conditions (1) and (2). Let J be any left ideal of $E(A)$. We claim J is finitely generated.

Order the path basis as described above. For any class $\varepsilon \in E(A)$, let $h(\varepsilon)$ be the largest basis element appearing with nonzero coefficient when ε is expressed in the path basis. Let $F^{\bullet}$ be the natural filtration on J inherited from the cohomology grading on $E(A)$. For $n > 0$ let $\mathcal{L}_n$ be the left ideal generated by $\{h(\varepsilon) : \varepsilon \in F^n J\}$. Let $\mathcal{L} = \bigcup_n \mathcal{L}_n$.

Conditions (1) and (2) guarantee the existence of a largest integer d such that the final edge in a path p of length d is not admissible. Then by Corollary 3.4, ε_q is decomposable for any anchored walk q of length $> d$. It follows that $\mathcal{L}/\mathcal{L}_d$ is finitely generated as a left ideal (if not, one could find anchored walks p and q with $p \vdash q$ and ε_p and ε_q algebraically independent); hence the ascending chain of left ideals $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \cdots$ stabilizes. The fact that J is finitely generated then follows by the standard Hilbert Basis argument.

Conversely, let $C = c_0 \cdots c_n$ be a circuit of length n in $\Gamma(A)$. First suppose vertex c_i has out-degree > 1, where $0 \leq i < n$. Let $v \neq c_{i+1}$ be a vertex such that $c_i \to v$ is an edge in $\Gamma(A)$. Let p be any anchored path of length m in $\Gamma(A)$ such that $p_m = c_{n-1}$. For $\ell \geq 0$, define $q_\ell = pC^\ell c_0 \cdots c_i v$ where C^ℓ indicates the circuit C is traversed ℓ times. Let J be the left ideal of $E(A)$ generated by $\{\varepsilon_{q_\ell} : \ell \geq 0\}$. We claim that J is not finitely generated.

If J is finitely generated, there exists $L > 0$ such that ε_{q_ℓ} is in the left ideal generated by $\varepsilon_{q_0}, \ldots, \varepsilon_{q_L}$ for all $\ell > L$. Fix $\ell_0 > L$. Then by Proposition 2.7 and Corollary 3.4, there exists a walk w and an index $0 \leq d \leq L$ such that $\varepsilon_{q_\ell} = \varepsilon_w \star \varepsilon_{q_d}$ and $q_\ell \sim q_d w$. Since $v \neq c_{i+1}$ and since $pC^d c_0 \cdots c_i$ is a prefix of q_ℓ, Lemma 2.10(2) implies q_d is an even-length walk. But by Lemma 2.10(3) and (4), $w_0 \otimes v = c_{i+2} \otimes c_{i+1}$ (where, if $i = n-1$, $c_{i+2} = c_1$) and $\deg(v) = \deg(c_{i+1})$, implying $v = c_{i+1}$, a contradiction. Thus if $\Gamma(A)$ contains a vertex of out-degree > 1 lying on a circuit, $E(A)$ is not left Noetherian.

Suppose instead that every vertex of C has out-degree 1 and C contains an edge $c_j \to c_{j+1}$ which is not admissible. Let K be the left ideal of $E(A)$ generated by ε_{q_i} for $i \geq 0$ where $q_i = pC^i c_0 \cdots c_{j-1}$ (if $j = 0$, then since $c_0 = c_n$ we take $q_i = pC^i c_0 \cdots c_{n-1}$). Since $c_j c_{j+1}$ is not admissible, and since c_j is the only successor of c_{j-1} in $\Gamma(A)$, it follows from Lemma 2.10(1) and Proposition 3.3 that $\varepsilon_w \star \varepsilon_{q_i} = 0$ for any admissible walk w. Thus K is an infinitely-generated trivial left ideal and $E(A)$ is not left Noetherian.

We omit the analogous proof for the right Noetherian case. $\square$

For the algebras A and B of Example 2.2, $E(A)$ and $E(B)$ are neither left nor right Noetherian. Comparing our graph-theoretic characterizations of GK dimension and the Noetherian property, we have the following immediate corollary.

Corollary 5.3. *Let A be a monomial k-algebra. If $E(A)$ is left or right Noetherian, then* $\operatorname{GKdim} E(A) \leq 1$. *If $E(A)$ is Noetherian then $\Gamma(A)$ consists of finitely many disjoint circuits and paths.*

6. Examples

The following example suggests that the $\operatorname{GKdim} E(A) = \infty$ case can be quite complicated; edges that are dense in one infinite walk need not be dense in another.

Example 6.1. Let $S = k\langle w, x, y, z, W, X, Y, Z, p, q\rangle$ be a free algebra and let I be the ideal generated by

$$\begin{array}{cccc} pqwxyz & wxyzp & xyzpqwx & YZpqwx \\ pqWXYZ & WXYZp & XYZpqWX & yzpqWX. \end{array}$$

Let A be the factor algebra $A = S/I$. The graph $\Gamma(A)$ has two components and is shown in Fig. 1. Admissible edges are indicated by solid arrows; dashed arrows are non-admissible edges. Vertex pq is common to two oriented circuits, so $\operatorname{GKdim} E(A) = \infty$. There are many infinite walks in $\Gamma(A)$. The walk

$$p \to wxyz \to pq \to wxyz \to pq \to \cdots$$

is anchored and $wxyz \to pq$ is dense in this walk since the even-length walk

$$wxyz \to pq \to wxyz \to pq \to wxyz$$

is equivalent to

$$z \to pqwxy \to xyz \to pqw \to wxyz.$$

However, the walk

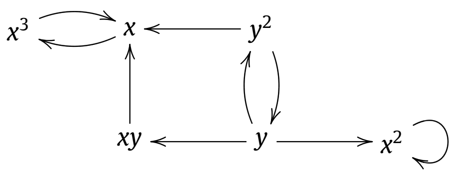
$$p \to wxyz \to pq \to WXYZ \to pq \to wxyz \to pq \to WXYZ \to \cdots$$

contains no dense edge. To see this, observe that the equivalent anchored walks corresponding to odd-length admissible extensions of $wxyz$ (and likewise of $WXYZ$) terminate on the circuit

$$yz \to pqwx \to YZ \to pqWX \to yz.$$

It follows that no even-length extension of $wxyz \to pq$ is admissible because condition (2) of Lemma 2.10 cannot be satisfied. By Theorem 3.6, $E(A)$ is not finitely generated.

Example 6.2. Let $A = k\langle x, y\rangle/\langle x^3 - x^2y, xy^2, y^3\rangle$ and observe $x^4 = 0$ in A. The degree–lexicographic ordering on monomials in $k\langle x, y\rangle$ with $x < y$ yields the associated monomial algebra $A' = k\langle x, y\rangle/\langle x^2y, xy^2, y^3, x^4\rangle$. Although $\dim E^i(A) \leq \dim E^i(A')$ for all i, one can check that equality does not always hold. The graph $\Gamma(A')$ is shown below. By Corollary 2.8, we have $\operatorname{GKdim} E(A') = 2$. It follows that $\operatorname{GKdim} E(A) \leq 2$.



$\Gamma(A')$

We leave to the reader the straightforward verification that $\operatorname{GKdim} E(A) > 1$; hence $\operatorname{GKdim} E(A) = 2$ by Bergman's gap theorem.

In many cases of interest, knowing $\operatorname{GKdim} E(A')$ provides little or no information about $\operatorname{GKdim} E(A)$. Consider the algebra

$$A = \frac{k\langle x, y, z\rangle}{\langle xy - z^2, zx - y^2, yz - x^2\rangle}.$$

The algebra A is a 3-dimensional Sklyanin algebra; hence $\operatorname{GKdim} E(A) = 0$. Using lexicographic ordering with $z > y > x$, the associated monomial algebra of A is

$$A' = \frac{k\langle x, y, z\rangle}{\langle z^2, zx, yz, y^3, zy^2, yxy, yx^3, y^2x^2, zyx^2\rangle}.$$

Constructing the CPS graph of A' reveals $\operatorname{GKdim} E(A') = \infty$.

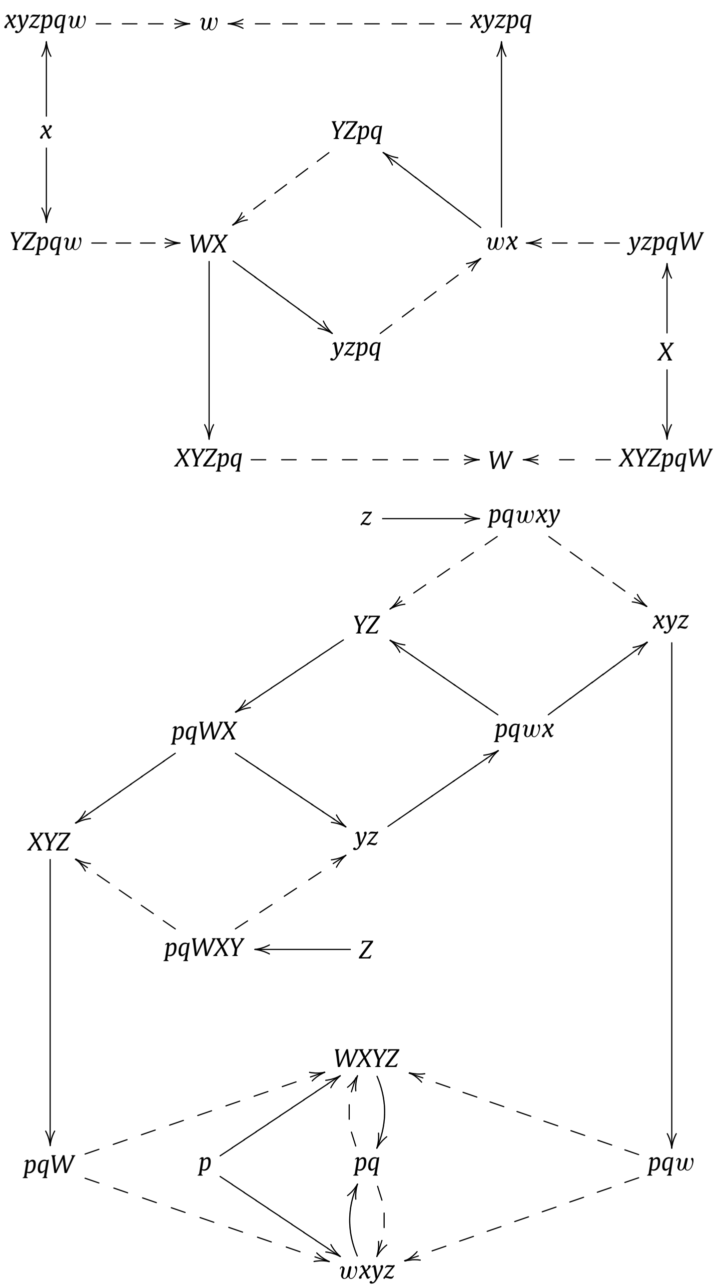


Fig. 1. The graph $\Gamma(A)$ for Example 6.1.

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