

Black-Hole Binary Initial Data

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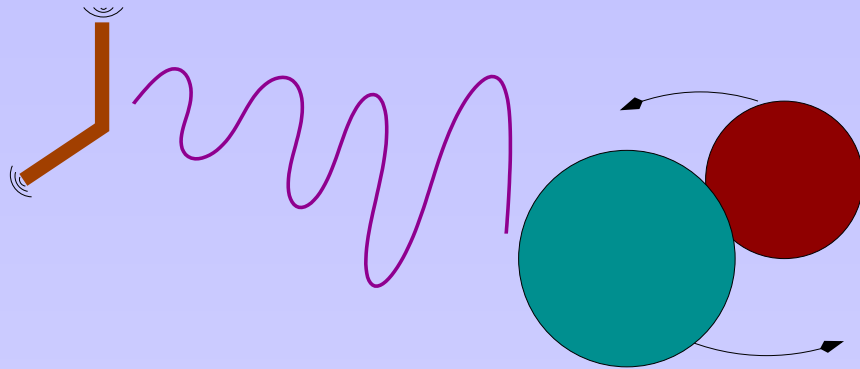
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Abstract

We will examine the current state of our efforts to generate astrophysically realistic initial data for black-hole binaries.

Collaborators: Harald Pfeiffer^[3] (Caltech), Jason D. Grigsby (WFU), & Matthew Caudill (WFU)

Motivation

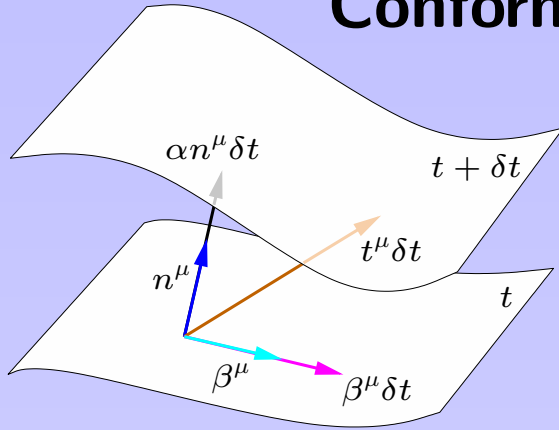


- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, . . .
- Available computed waveforms should increase chance of detecting collision events.

Quasiequilibrium Binary Data

- General Relativity doesn't permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.
- ★ Quasiequilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

Conformal Thin-Sandwich Decomposition



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K$$

Hamiltonian Const. $\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$

Momentum Const. $\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^iK + \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$

Const. Tr(K) eqn. $\tilde{\nabla}^2(\psi^7\tilde{\alpha}) - (\psi^7\tilde{\alpha})\left[\frac{1}{8}\psi\tilde{R} + \frac{5}{12}\psi^5K^2 + \frac{7}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} - \psi^5\beta^i\tilde{\nabla}_iK\right]$
 $= -2\pi\psi^5K(\rho + 2S) - \psi^5\partial_tK$

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

Constrained vars : ψ , β^i , and $\tilde{\alpha} \equiv \psi^{-6}\alpha$

Freely specified : $\tilde{\gamma}_{ij}$ $\tilde{u}^{ij} \equiv -\partial_t\tilde{\gamma}^{ij}$
 K *and* ∂_tK

$$\text{Quasiequilibrium} \Rightarrow \begin{cases} \partial_t\tilde{\gamma}^{ij} = 0 \\ \partial_tK = 0 \end{cases}$$

Equations of Quasiequilibrium

$$\left. \begin{array}{l} \text{Ham. \& Mom. const.} \\ \text{eqns., \& Const Tr}(K) \\ \text{eqn. from Conf. TS} \\ + \tilde{u}^{ij} = \partial_t K = 0 \end{array} \right\} \Rightarrow \text{Eqns. of} \\ \text{Quasiequilibrium}$$

With $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 0$, these equations have been widely used to construct binary neutron star initial data [1, 6, 2, 7].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{r \rightarrow \infty} = 1 \quad \beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \rightarrow \infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. ($\Rightarrow \Omega$)

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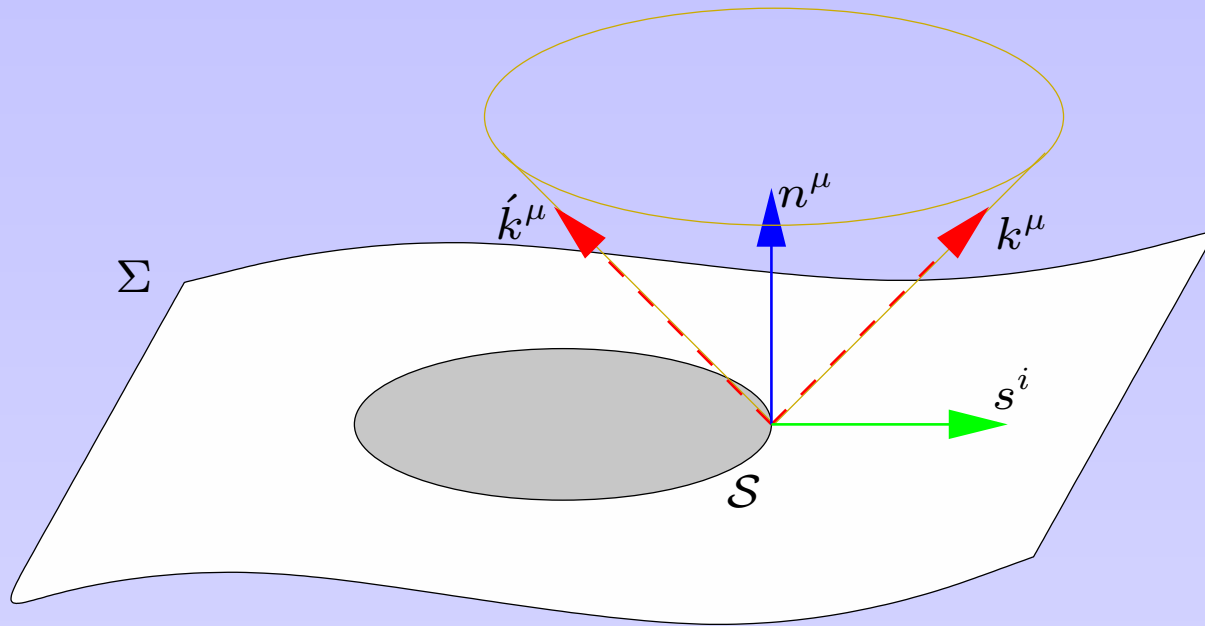
Binary black hole initial data require:

- ★ a means for choosing the angular velocity of the orbit Ω .
- ★ *with excision*, inner boundary conditions are needed for ψ , β^i , and $\tilde{\alpha}$.

Gourgoulhon, Grandclément, & Bonazzola[4, 5]: Black-hole binaries with $\tilde{\gamma}_{ij} = f_{ij}$ & $K = 0$, “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require *constraint violating* regularity condition imposed on inner boundaries!

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of S embedded in spacetime

$$\Sigma_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

$$\hat{\Sigma}_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

Extrinsic curvature of S embedded in Σ

$$H_{ij} \equiv \frac{1}{2} h_i^k h_j^\ell \mathcal{L}_s \gamma_{kl}$$

$$\Sigma_{ij} = \frac{1}{\sqrt{2}} (H_{ij} - J_{ij})$$

$$\hat{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H_{ij} + J_{ij})$$

Projections of K_{ij} onto S

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

$$J_i \equiv h_i^k s^\ell K_{kl}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (H - J)$$

$$\hat{\theta} \equiv h^{ij} \hat{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H + J)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$$

$$\hat{\sigma}_{ij} \equiv \hat{\Sigma}_{ij} - \frac{1}{2} h_{ij} \hat{\theta}$$

AH and QE Conditions on the Inner Boundary

The quasiequilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary \mathcal{S} is a (MOTS):
marginally outer-trapped surface

$$\rightarrow \theta = 0$$

2. The horizons are in quasiequilibrium:

$$\rightarrow \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S}$$

Raychaudhuri's equation implies that MOTS initially evolves along k^μ .

$$\mathcal{L}_k \theta = \frac{1}{2} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu = 0$$

3. The time evolution vector is null on \mathcal{S} :

$$\rightarrow t^\mu k_\mu|_{\mathcal{S}} = 0$$

$$\left. \begin{aligned} k^\mu &\equiv \frac{1}{\sqrt{2}}(n^\mu + s^\mu) \\ t^\mu &= \alpha n^\mu + \beta^\mu \end{aligned} \right\} \Rightarrow \alpha|_{\mathcal{S}} = \beta^i s_i|_{\mathcal{S}} \equiv \beta_\perp|_{\mathcal{S}}$$

AH/Quasiequilibrium Boundary Conditions

$$\theta = \frac{\psi^{-2}}{\sqrt{2}} \left[\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4\tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right]$$

$$\begin{aligned} \sigma_{ij} = & \frac{1}{\sqrt{2}} (H_{ij} - \frac{1}{2} h_{ij} H) \left(1 - \frac{\beta_{\perp}}{\alpha} \right) \\ & - \frac{1}{\sqrt{2}} \frac{\psi^4}{\alpha} \left\{ \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{jl} \tilde{u}^{kl} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{kl} \tilde{u}^{kl}] \right\} \end{aligned}$$

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$$\tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)$$

$$\beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_{\parallel}^i$$

$$0 = \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k$$

The boundary condition on the lapse is not fixed by QE assumptions. It has been shown to be associated with the initial temporal gauge choice and it taken to be freely specifiable.

Defining the Spin of the Black Hole

The spin parameters β_{\parallel}^i can be defined by demanding that the time vector associated with quasiequilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^{\mu} \propto (n^{\mu} + s^{\mu}) \implies k^{\mu} = [1, \alpha s^i - \beta^i]$$

This vector k^{μ} is null for any choice of α & β^i .

In the frame where a black hole is not spinning, the null time vector has components $t^{\mu} = [1, \vec{0}]$.

Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta_{\parallel}^i = 0$$

Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi} \right)^i \implies \beta_{\parallel}^i = \Omega \xi^i$$

$$\xi^i \approx \left(\frac{\partial}{\partial \phi} \right)^i \quad \& \quad \tilde{D}_{(i} \xi_{j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \xi^k = 0$$

The Orbital Angular Velocity

- For a given choice of the Lapse BC, $\tilde{\gamma}_{ij}$ and K , we are still left with a family of solutions parameterized by the orbital angular velocity Ω .
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasiequilibrium.

GGB[4, 5] have suggested a way to pick the quasiequilibrium value of Ω :

Ω is chosen as the value for which the ADM energy E_{ADM} equals the Komar mass M_{K} .

Komar
mass

$$M_{\text{K}} = \frac{1}{4\pi} \oint_{\infty} \gamma^{ij} (\bar{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j$$

Acceptable definition of the mass
only for stationary spacetimes.

ADM
energy

$$E_{\text{ADM}} = \frac{1}{16\pi} \oint_{\infty} \gamma^{ij} \bar{\nabla}_k (\mathcal{G}_i^k - \delta_i^k \mathcal{G}) d^2 S_j$$

Acceptable definition of the mass
for arbitrary spacetimes.

$$\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$$

Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0$$

$$\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j (\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K$$

$$\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[\frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} A_{ij} A^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \theta = 0$$

$$\beta^i|_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i|_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i|_S + \Omega \xi^i|_S & \text{irrotation} \end{cases}$$

$$\sigma_{ij} = 0 \\ t^\mu k_\mu = 0$$

$\alpha|_S =$ unspecified by QE

$$\psi|_{r \rightarrow \infty} = 1$$

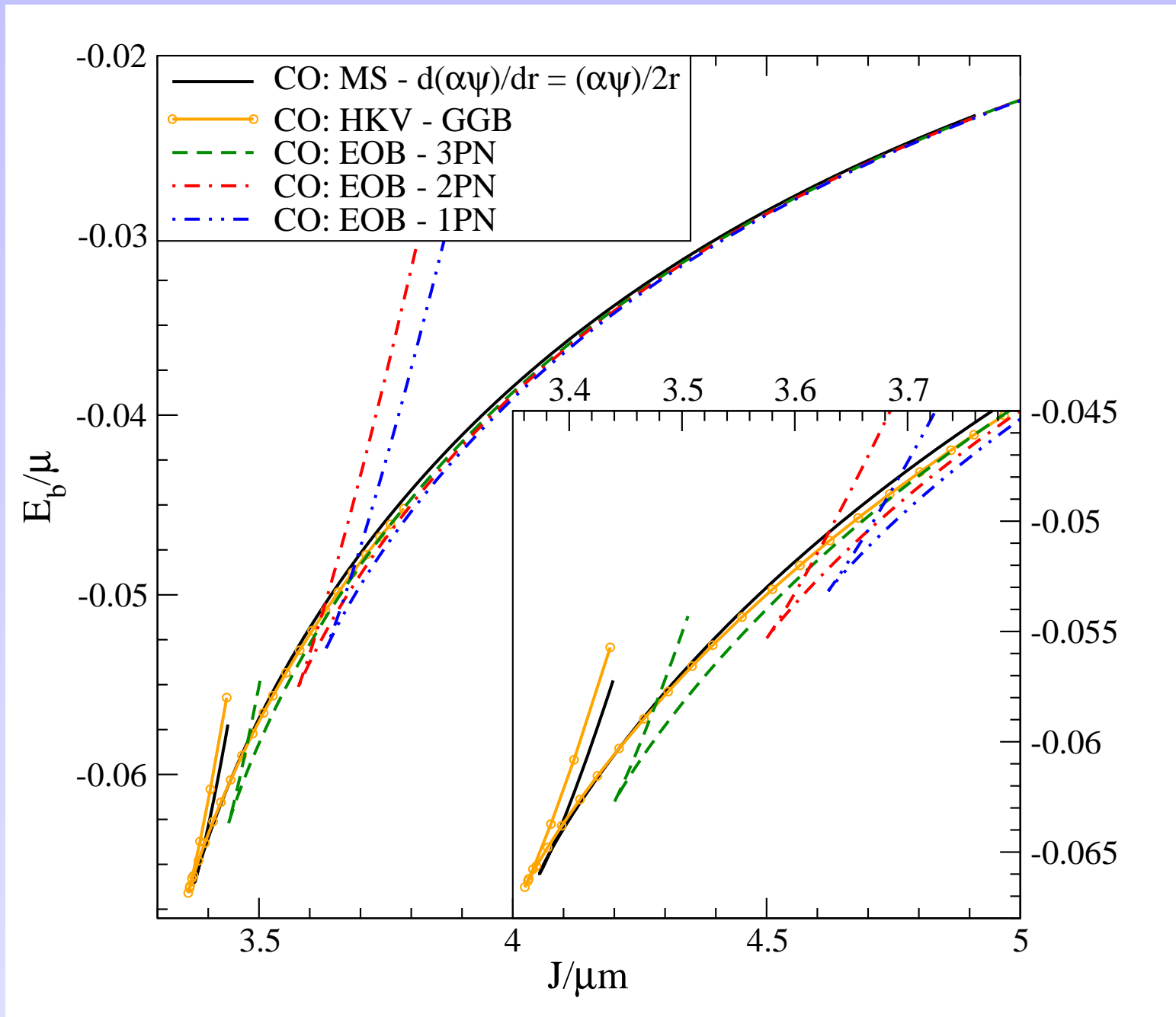
$$\beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i$$

$$\alpha|_{r \rightarrow \infty} = 1$$

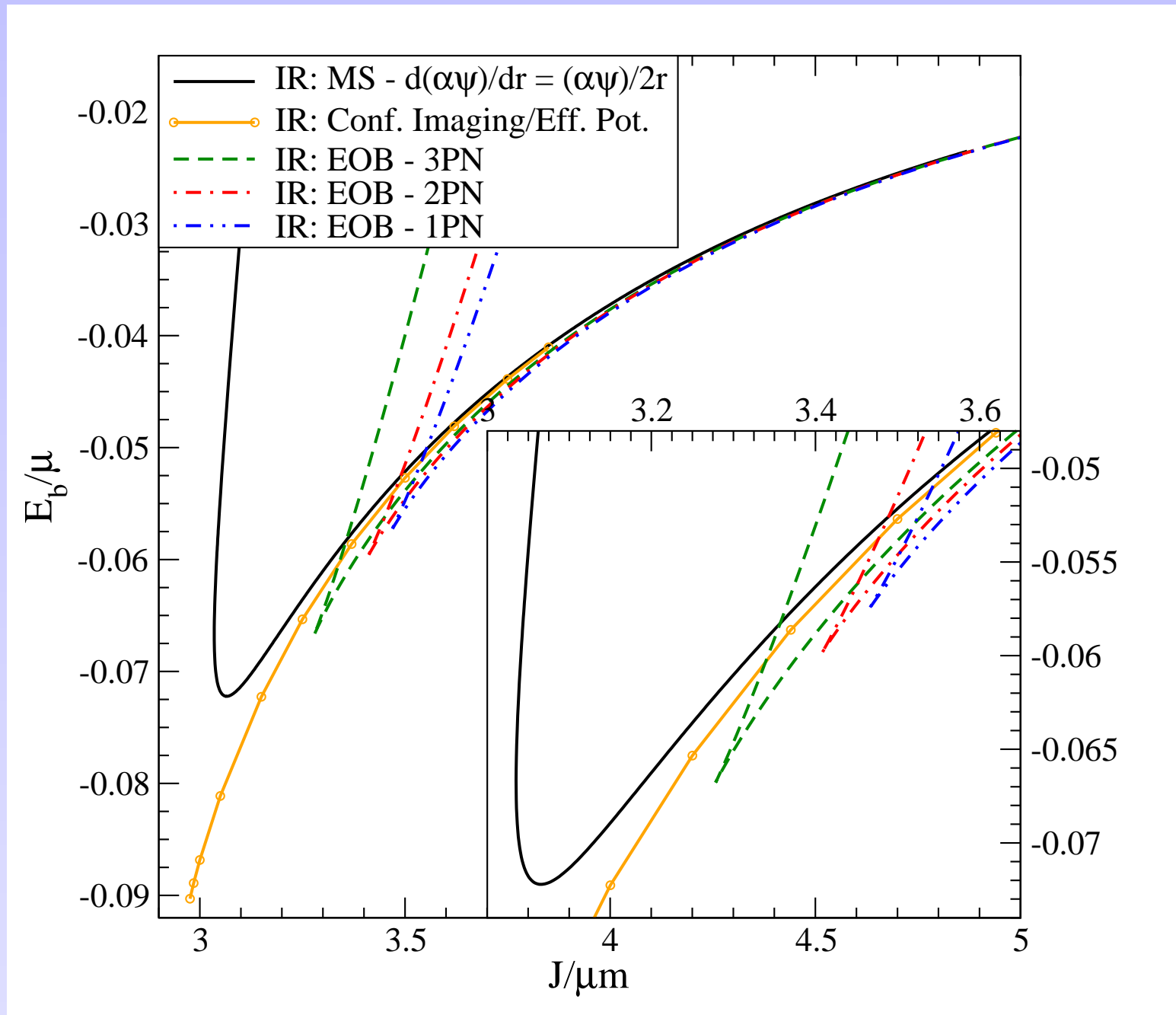
The only remaining freedom in the system is the choice of the lapse boundary condition, the initial spatial and temporal gauge, and the initial dynamical (“wave”) content found in

$\alpha|_S$, $\tilde{\gamma}_{ij}$ and K .

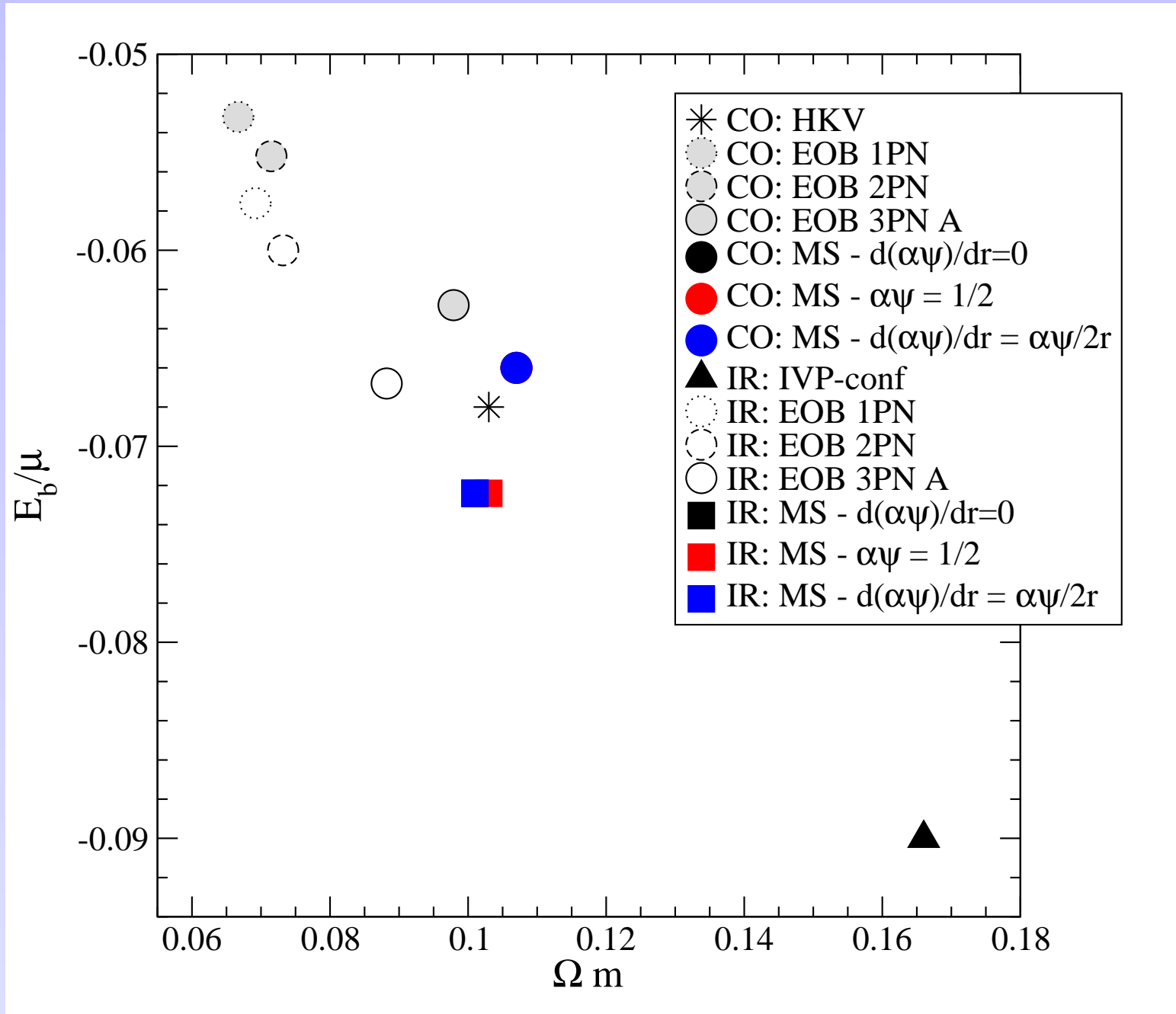
Corotating; Maximal Slice; Comparison; E_b/μ vs $J/\mu m$



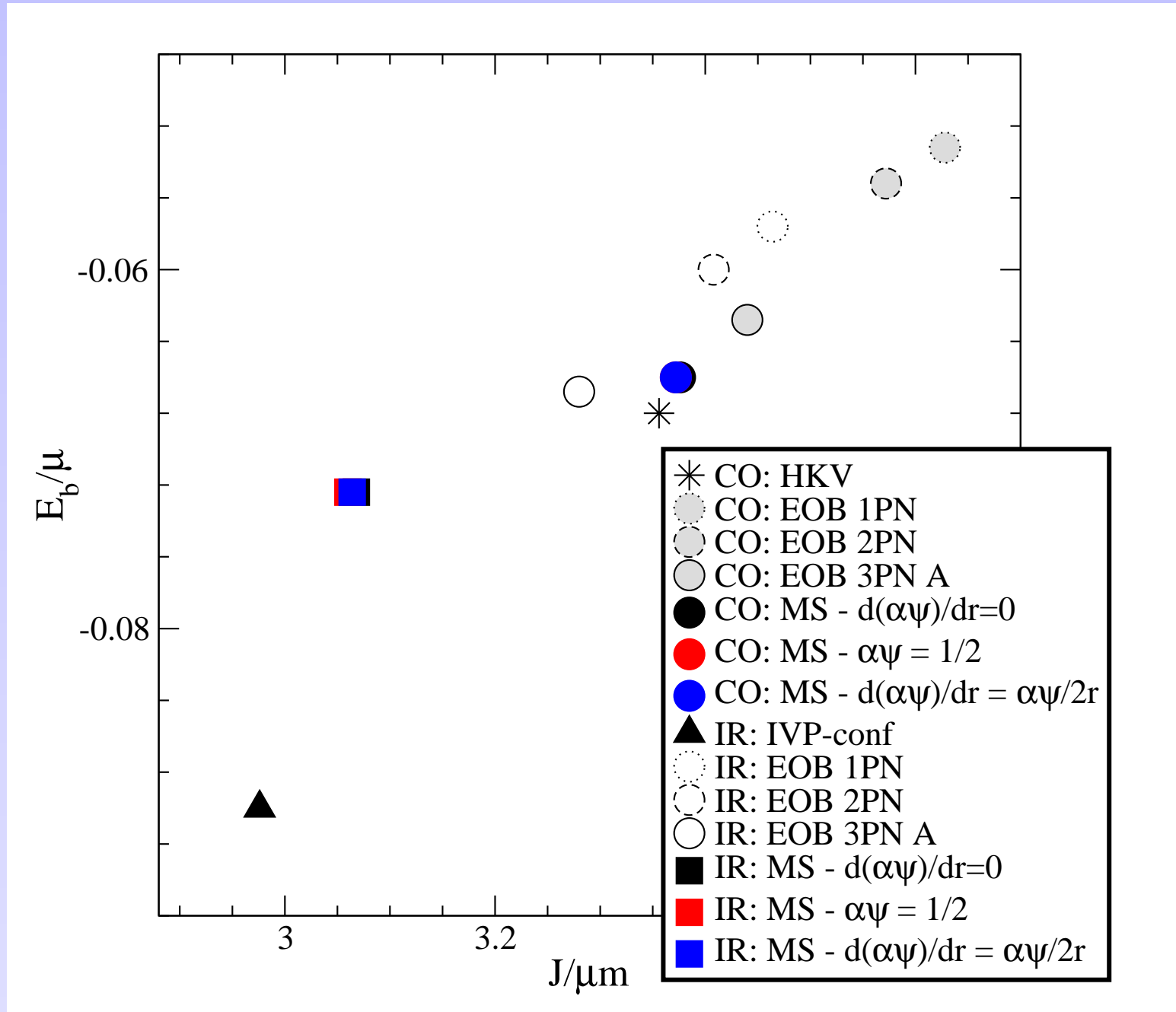
Irrotational; Maximal Slice; Comparison; E_b/μ vs $J/\mu m$



Maximal Slice; Comparison of ISCO; E_b/M_{irr} vs ΩM_{irr}



Maximal Slice; Comparison of ISCO; E_b/M_{irr} vs J/M_{irr}^2



Open Questions

- * Is the physics of the corotating and irrotational models correct?
 - Do the corotating black holes have the correct angular momentum?
 - Is the angular momentum of the irrotational holes nearly zero?
- * Is the orbital angular velocity being chosen correctly?
- Can we use this approach to construct binaries in elliptical orbits?

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