Dynamic Model: Practice Problem Key
Intermediate Macroeconomics
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Question 1

a) A competitive equilibrium in the dynamic, two period classical model with a capital market is a price (r^*) and allocations ($c_1^*, c_2^*, S^*, K^*, Y^{S*}$) such that the following conditions hold:

1) Given the price (r^*), the representative household solves

$$\max_{c_1,c_2,S} U(c_1,c_2) \quad s.t. \quad c_1 + S = Y_1$$

$$c_2 = Y_2 + (1+r^*)S$$

$$Y_1,\ Y_2 \text{ given}$$

2) Given the price (r^*), the representative firm solves

$$\max_{Y^S,K} Y^S - (1+r^*)K \quad s.t. \quad Y^S = f(K)$$

3) Markets clear

$$S^* = K^*$$

b) Savings supply is derived from the household's problem.

Rewriting the representative household's maximization problem in terms of S, we have the following:

$$\max_S \ \ Y_1 - S + \beta(Y_2 + (1+r)S)$$

Using the fact that we know $Y_1 = 25,\ Y_2 = 0,\ \beta = \frac{4}{5}$, the above can be written as:

$$\max_S \ \ 25 - S + \frac{4}{5}(1+r)S$$

$$\mathcal{L} = 25 - S + \frac{4}{5}(1+r)S$$

$$\frac{\partial \mathcal{L}}{\partial S} = -1 + \frac{4}{5}(1+r) = 0$$

$$1 = \frac{4}{5}(1+r)$$

$$\Rightarrow r^* = \frac{1}{4}$$

Notice that since the S term has disappeared, we will have a perfectly elastic supply of savings.

Investment demand is derived from the firm's problem:

$$\max_{Y^S,K} \; Y^S - (1+r)K \quad s.t. \quad Y^S = 10K^{\frac{1}{2}}$$

This can be written as the following:

$$\max_{K} \; 10K^{\frac{1}{2}} - (1+r)K$$

$$\mathcal{L} = 10K^{\frac{1}{2}} - (1+r)K$$

$$\frac{\partial \mathcal{L}}{\partial K} = 5K^{\frac{-1}{2}} - (1+r) = 0$$

$$\Rightarrow K = \frac{25}{(1+r)^2}$$

We now can combine the supply of savings and demand for investment to obtain an equilibrium in the capital market.

From the the representative household's problem, we have $r^* = \frac{1}{4}$. We can plug this value into our expression for K, $K = \frac{25}{(\frac{5}{4})^2}$. Thus, $K^* = 16$. From the market clearing condition, $S^* = 16$ as well.

To solve for Y^{S^*}, plug the equilibrium value of capital, K^*, into the representative firm's production function, $Y^{S^*} = f(K^*) = 10(16)^{\frac{1}{2}} = 40$.

Using the representative household's period 1 and period 2 budget constraints, ${C_1}^* = Y_1 - S^* = 25 - 16 = 9$ and ${C_2}^* = Y_2 + (1+r^*)S^* = 0 + (1+\frac{1}{4})16 = 20$.

Question 2

a) The savings supply curve is derived from the household's problem, which is written as follows:

$$\max_{c_1,c_2,S} \ln \; c_1 + \frac{3}{4} \ln \; c_2 \quad s.t. \quad c_1 + S = 50$$
$$c_2 = (1+r)S$$

Rewriting the representative household's maximization problem in terms of S and solving for S, we have the following:

$$\max_{S} \; \ln(50-S) + \frac{3}{4}\ln((1+r)S)$$

$$\mathcal{L} = \ln(50-S) + \frac{3}{4}\ln((1+r)S)$$

$$\frac{\partial \mathcal{L}}{\partial S} = -\frac{1}{50-S} + \frac{3}{4}\frac{(1+r)}{(1+r)S} = 0$$

$$\frac{1}{50-S} = \frac{3}{4S}$$

$$4S = 150 - 3S$$

$$\Rightarrow S = \frac{150}{7} = 21.43$$

Notice we have a perfectly inelastic savings supply curve, so the equilibrium quantity of savings S^* is immediately determined: $S^* = 21.43$.

Investment demand is derived from the firm's problem:

$$\max_{Y^S, K} \; Y^S - (1+r)K \quad s.t. \quad Y^S = 20K^{\frac{1}{2}}$$

This can be written as the following:

$$\max_{K} \; 20K^{\frac{1}{2}} - (1+r)K$$

$$\mathcal{L} = 20K^{\frac{1}{2}} - (1+r)K$$

$$\frac{\partial \mathcal{L}}{\partial K} = 10K^{\frac{-1}{2}} - (1+r) = 0$$

$$K^{\frac{-1}{2}} = \frac{(1+r)}{10}$$

$$\Rightarrow K = \frac{100}{(1+r)^2}$$

We now can combine the supply of savings and demand for investment to obtain an equilibrium in the capital market. From the market clearing condition, $S^* = K^* = 21.43$. Again, using the market clearing condition, we can equate the saving supply curve and the investment demand curve to find the equilibrium interest rate r^*:

$$\frac{150}{7} = \frac{100}{(1+r^*)^2}$$

$$(1+r^*)^2 = \frac{700}{150}$$

$$\Rightarrow r^* = \left(\frac{700}{150}\right)^{\frac{1}{2}} - 1 = 1.16$$

To solve for Y^{S^*}, plug the equilibrium value of capital, K^*, into the representative firm's production function, $Y^{S^*} = f(K^*) = 20\left(\frac{150}{7}\right)^{\frac{1}{2}} = 92.58$.

Using the representative household's period 1 and period 2 budget constraints, $C_1{}^* = 50 - S^* = 50 - \frac{150}{7} = 28.57$ and $C_2{}^* = (1+r^*)S^* = (1+1.16)\frac{150}{7} = 46.29$.

b) The savings supply curve is derived from the household's problem, which is written as follows:

$$\max_{c_1,c_2,S} \ln\ c_1 + \frac{3}{4}\ln\ c_2 \quad s.t. \quad c_1 + S = 50(1-\tau)$$
$$c_2 = (1+r)S$$

Rewriting the representative household's maximization problem in terms of S and solving for S, we have the following:

$$\max_S\ \ln(50(1-\tau) - S) + \frac{3}{4}\ln((1+r)S)$$

$$\mathcal{L} = \ln(50(1-\tau) - S) + \frac{3}{4}\ln((1+r)S)$$

$$\frac{\partial \mathcal{L}}{\partial S} = -\frac{1}{50(1-\tau) - S} + \frac{3}{4}\frac{(1+r)}{(1+r)S} = 0$$

$$\frac{1}{50(1-\tau) - S} = \frac{3}{4S}$$

$$4S = 150(1-\tau) - 3S$$

$$\Rightarrow S = \frac{150}{7}(1-\tau) = 21.43(1-\tau)$$

Again, we have a perfectly inelastic savings supply curve, but it has shifted to the left due to the tax. The equilibrium quantity of savings is $S^* = 21.43(1-\tau)$.

Nothing has changed on the firm's side of the market, so

$$K = \frac{100}{(1+r)^2}$$

We now can combine the supply of savings and demand for investment to obtain an equilibrium in the capital market. From the market clearing condition, $S^* = K^* = 21.43(1-\tau)$. Again, using the market clearing condition, we can equate the saving supply curve and the investment demand curve to find the equilibrium interest rate r^*:

$$\frac{150}{7}(1-\tau) = \frac{100}{(1+r^*)^2}$$

$$(1+r^*)^2 = \frac{700}{150(1-\tau)}$$

$$\Rightarrow r^* = \left(\frac{700}{150(1-\tau)}\right)^{\frac{1}{2}} - 1$$

Notice

$$\frac{\partial r^*}{\partial \tau} = \frac{1}{2}\left(\frac{700}{150}\right)^{\frac{1}{2}}(1-\tau)^{-\frac{3}{2}} > 0,$$

which means as the tax rate τ increases the equilibrium interest rate r^* increases. This comes from the fact that increasing taxes shifts the savings supply curve to the left in the capital market.

To solve for Y^{S*}, plug the equilibrium value of capital, K^*, into the representative firm's production function, $Y^{S*} = f(K^*) = 20(21.43(1-\tau))^{\frac{1}{2}} = 92.58(1-\tau)^{\frac{1}{2}}$.

Using the representative household's period 1 and period 2 budget constraints, ${C_1}^* = 50(1-\tau) - 21.43(1-\tau) = 28.57(1-\tau)$ and ${C_2}^* = (1 + \left(\frac{700}{150(1-\tau)}\right)^{\frac{1}{2}} - 1)(21.43(1-\tau)) = 46.29(1-\tau)^{\frac{1}{2}}$.

Question 3

a) Given the form of the household's utility function, the household derives the same amount of utility when consuming in period 1 and period 2. Imagine if the household chooses to consume 10 units of consumption. Whether the household consumes these units in period 1 or period 2 does not change the amount of utility, which means the household will not necessarily smooth its consumption, e.g. $U(10,0) = U(0,10)$. Changing the household's utility function to $U(c_1, c_2) = \ln\ c_1 + \ln\ c_2$ would provide an incentive for the household to smooth its consumption, or at least consume positive amounts in each period, because, as $c_1 \to 0$ or $c_2 \to 0$, $U(c_1, c_2) \to -\infty$.

b) The savings supply curve is derived from the household's problem, which is written as follows:

$$\max_{c_1, c_2, S} \ c_1 + \beta c_2 \quad s.t. \quad c_1 + S = Y_1$$

$$c_2 = Y_2 + (1+r)S$$

Rewriting the representative household's maximization problem in terms of S and solving for S, we have the following:

$$\max_{S} \ Y_1 - S + \beta[Y_2 + (1+r)S]$$

$$\Rightarrow \mathcal{L} = Y_1 - S + \beta[Y_2 + (1+r)S]$$

$$\Rightarrow \mathcal{L} = 15 - S + \frac{1}{2}[0 + (1+r)S]$$

$$\frac{\partial \mathcal{L}}{\partial S} = -1 + \frac{1}{2}(1+r) = 0$$

$$(1+r) = 2$$

$$\Rightarrow r^* = 1$$

Investment demand is derived from the firm's problem:

$$\max_{Y^S, K} \ Y^S - (1+r)K \quad s.t. \quad Y^S = K^{\frac{1}{2}}$$

This can be written as the following:

$$\max_{K} \quad K^{\frac{1}{2}} - (1+r)K$$

$$\Rightarrow \mathcal{L} = K^{\frac{1}{2}} - (1+r)K$$

$$\frac{\partial \mathcal{L}}{\partial K} = \frac{1}{2}K^{\frac{-1}{2}} - (1+r) = 0$$

$$K^{\frac{-1}{2}} = 2(1+r)$$

$$\Rightarrow K^* = \frac{1}{4(1+r)^2} = \frac{1}{16}$$

We now can combine the supply of savings and demand for investment to obtain an equilibrium in the capital market. From the market clearing condition, $S^* = K^* = \frac{1}{16}$.

To solve for Y^{S^*}, plug the equilibrium value of capital, K^*, into the representative firm's production function, $Y^{S^*} = f(K^*) = \left(\frac{1}{16}\right)^{\frac{1}{2}} = \frac{1}{4}$.

Using the representative household's period 1 and period 2 budget constraints, ${C_1}^* = Y_1 - S^* = 15 - \frac{1}{16} = 14\frac{15}{16}$ and ${C_2}^* = Y_2 + (1+r^*)S^* = 0 + (1+1)\frac{1}{16} = \frac{2}{16} = \frac{1}{8}$.