Physics 310/610 – Cosmology

Solution Set B

1. **[20] The sun is d = 1.000 AU from the Earth.**

   (a) [4] Calculate the brightness of the Sun F at Earth in terms of $L_\odot$ and d.

The brightness is simply calculated from the formula $L = 4\pi d^2 F$, so we have

$$F = \frac{L_\odot}{4\pi d^2} = \frac{3.828\times10^{26}\ \text{W}}{4\pi\left(1.496\times10^{11}\ \text{m}\right)^2} = 1368\ \text{W/m}^2\ .$$

   (b) [4] The Earth intercepts the light coming from the Sun if it is intersects its cross-section, a circle of radius R. What is the total power falling on the Earth?

The *cross-section* area of the Earth, or the size of the Earth as would be viewed from the Sun, is simply πR^2. Therefore, the total power falling on the Earth is

$$P = F\pi R^2 = \frac{L_\odot R^2}{4d^2} = \frac{\left(3.828\times10^{26}\ \text{W}\right)\left(6.370\times10^{6}\ \text{m}\right)^2}{4\left(1.496\times10^{11}\ \text{m}\right)^2} = 1.743\times10^{17}\ \text{W}\ .$$

   (c) [7] Assume the Earth is a perfect blackbody, so that all the light you found in part (a) is absorbed. Assume that all of this power is then reradiated at a constant temperature T (but this time from the entire surface of the Earth). Find the equilibrium temperature T in terms of $L_\odot$ and d. Notice that R, the area of the Earth, cancels out.

The power radiated would just be $P = 4\pi R^2\sigma T^4$, so solving for T, we have

$$T^4 = \frac{P}{4\pi R^2\sigma} = \frac{L_\odot R^2}{4d^2\cdot 4\pi R^2\sigma} = \frac{L_\odot}{16\pi\sigma d^2},$$

$$T = \left(\frac{L_\odot}{16\pi\sigma d^2}\right)^{1/4}.$$

   (d) [5] Calculate T in K for d = 1.000 AU. The number should come out about right.

Substituting into the equation we just found, we have:

$$T = \left(\frac{3.828\times10^{26}\ \text{W}}{16\pi\left(5.670\times10^{-8}\,\text{W m}^{-2}\,\text{K}^{-4}\right)\left(1.496\times10^{11}\ \text{m}\right)^2}\right)^{1/4} = \left(6.001\times10^{9}\ \text{K}^4\right)^{1/4} = 278\,\text{K}.$$

This is about 5°C, which is actually just a bit below the actual temperature.

2. [10] Which is hotter, heaven or hell? Use the following data to find out:

(a) Isaiah 30:26 says, “Moreover the light of the moon shall be as the light of the sun, and the light of the sun shall be sevenfold, as the light of seven days.” So Moon plus Sun is 1 + 7×7 times the regular luminosity. Based on this, redo problem (1) to find the temperature of heaven.

Since all the calculations were done symbolically in the previous problem, we can simply replace $L_\odot$ by $50L_\odot$ to yield

$$T_{\text{heaven}} = \left(\frac{50L_\odot}{16\pi\sigma d^2}\right)^{1/4} = \left(\frac{50\times 3.828\times 10^{26}\ \text{W}}{16\pi\left(5.670\times 10^{-8}\,\text{W m}^{-2}\ \text{K}^{-4}\right)\left(1.496\times 10^{11}\ \text{m}\right)^2}\right)^{1/4} = \left(3.001\times 10^{11}\ \text{K}^4\right)^{1/4}$$

$$= 740\,\text{K}.$$

This is about 467° C, so pretty hot!

(b) Revelation 21:8 says, “But the cowardly, the unbelieving … will be consigned to the fiery lake of burning sulfur.” Look up the melting and boiling point of sulfur and deduce upper and lower bounds on the temperature of hell. Which is hotter?

According to Wikipedia, sulfur melts at 388 K and boils at 718 K, so we have

$$388\ \text{K} < T_{\text{hell}} < 718\ \text{K}.$$

Since all of this range is below the calculated temperature of heaven, we conclude that heaven is hotter than hell.

Graduate Problems – Only do these problems if you are in PHY 610

3. **[10] The total energy density can be found by doing the following integral over the wave number k:**

$$u = 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{E}{e^{E/k_BT}-1}$$

Show that this can be written in terms of an integral over the wavelength or frequency, so $u = \int_0^\infty u_\nu(\nu)\,d\nu = \int_0^\infty u_\lambda(\lambda)\,d\lambda$, and find the form of $u_\nu(\nu)$ and $u_\lambda(\lambda)$. Also, perform the integral (feel free to use software to assist you if necessary).

We rewrite the integral in spherical coordinates, so we have

$$u = \frac{2}{(2\pi)^3}\int_0^\infty k^2 dk \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \frac{E}{e^{E/k_BT}-1} = \frac{2\cdot 2\pi\cdot 2}{(2\pi)^3}\int_0^\infty k^2 dk \frac{E}{e^{E/k_BT}-1}$$

We now change variables. First we change it to the frequency, which is related to the wave number by $k = 2\pi/\lambda$, and then using $\lambda = c/\nu$, we have $k = 2\pi\nu/c$. We also recall that $E = h\nu = hc/\lambda$, so we have

$$u = \frac{8\pi}{8\pi^3}\int_0^\infty \left(\frac{2\pi\nu}{c}\right)^2 d\left(\frac{2\pi\nu}{c}\right)\frac{h\nu}{e^{h\nu/k_BT}-1} = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/k_BT}-1},$$

$$u = \frac{8\pi}{8\pi^3}\int_0^\infty \left(\frac{2\pi}{\lambda}\right)^2 d\left(\frac{2\pi}{\lambda}\right)\frac{hc/\lambda}{e^{hc/\lambda k_BT}-1} = 8\pi hc\int_\infty^0 \lambda^{-2}\left(-\lambda^{-2}d\lambda\right)\frac{\lambda^{-1}}{e^{hc/\lambda k_BT}-1} = 8\pi hc\int_0^\infty \frac{d\lambda}{\lambda^5}\frac{1}{e^{hc/\lambda k_BT}-1}.$$

A word is perhaps in order about the sudden reversal of the integrals in the second case. The reason the limits change is because $k = 0$ corresponds to $\lambda = \infty$, and vice versa. Switching the limits of the integral then introduces a minus sign, cancelling the minus sign that came from the variable change. We therefore have

$$u_\nu(\nu) = \frac{8\pi h\nu^3}{c^3\left(e^{h\nu/k_BT}-1\right)},\quad u_\lambda(\lambda) = \frac{8\pi hc}{\lambda^5\left(e^{hc/\lambda k_BT}-1\right)}.$$

To perform the integral, it is probably easiest to start with the frequency expression, and define $x = h\nu/k_BT$. We therefore have

$$u = \frac{8\pi h}{c^3}\int_0^\infty \left(\frac{k_BT}{h}x\right)^3 d\left(\frac{k_BT}{h}x\right)\frac{1}{e^x-1} = \frac{8\pi h}{c^3}\left(\frac{k_BT}{h}\right)^4\int_0^\infty \frac{x^3dx}{e^x-1} = \frac{8\pi(k_BT)^4}{(hc)^3}6\cdot\frac{\pi^4}{90} = \frac{8\pi^5(k_BT)^4}{15h^3c^3}$$

$$= \frac{\pi^2(k_BT)^4}{15\hbar^3c^3}.$$

This has been simplified slightly with the substitution $h = 2\pi\hbar$. The x integral can easily be worked out with a variety of techniques, or you can let Maple do it:

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> integrate(x^3/(exp(x)-1),x=0..infinity);
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4. **[5] To find the total amount of flux crossing the *z* = 0 plane, an additional factor of v_z must be incorporated into the previous integral, where v_z is the velocity in the *z*-direction, and the integral must be restricted to just $v_z > 0$. Perform the relevant integral, and derive the Stefan-Boltzmann law.**

Light moves at *c*, the speed of light, and the *z*-component would be $c\cos\theta$. We only want those components that are in the positive *z*-direction, so we need $\cos\theta > 0$, which restricts us to $\theta < \frac{1}{2}\pi$. We therefore would have

$$F = u = 2\int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{Ev_z}{e^{E/k_BT}-1}\theta(v_z) = \frac{2c}{(2\pi)^3}\int_0^\infty k^2 dk \int_0^{2\pi} d\phi \int_0^{\frac{1}{2}\pi} \sin\theta\cos\theta d\theta \frac{E}{e^{E/k_BT}-1}$$

$$= \frac{2c \cdot 2\pi \cdot \frac{1}{2}}{(2\pi)^3}\int_0^\infty k^2 dk \frac{E}{e^{E/k_BT}-1} = \tfrac{1}{4}cu.$$

The penultimate expression is obviously just a slightly modified version of the energy density formula, so we have

$$F = \frac{\pi^2 (k_B T)^4}{60\hbar^3 c^2}.$$

This is the Stefan-Boltzmann law.