Physics 310/610 – Cosmology

Solution Set X

1. **[10] One of the least luminous stars is the obscure red dwarf 2MASS J0523-1403. It has a luminosity of $L = 1.26\times10^{-4}\,L_\odot$ and a mass probably around $M = 0.080\,M_\odot$.**
   - **(a) Assuming the star is undergoing nuclear fusion, $4\,{}^1\text{H} + 2e^- \to {}^4\text{He} + 2\nu + 26.73\text{ MeV}$, what mass of ^{1}H is being consumed every second to keep this star powered?**

The total power is

$$L = 1.26\times10^{-4}\,L_\odot = \left(1.26\times10^{-4}\right)\left(3.828\times10^{26}\,\text{W}\right) = 4.823\times10^{22}\text{ W}$$

Each interaction results in 26.73 MeV of energy, so to produce this many watts would require a rate for this reaction of

$$\Gamma = \frac{L}{E} = \frac{4.823\times10^{22}\text{ W}}{\left(26.73\times10^{6}\text{ eV}\right)\left(1.602\times10^{-19}\text{ J/eV}\right)} = 1.126\times10^{34}\text{ s}^{-1}\,.$$

The mass used up is essentially the mass of four hydrogen atoms, which have a mass of 1.6727×10^{-27} kg, so the rate at which mass is consumed is

$$\frac{dM}{dt} = 4\Gamma m_H = 4\left(1.126\times10^{34}\text{ s}^{-1}\right)\left(1.6727\times10^{-27}\text{ kg}\right) = 7.535\times10^{7}\text{ kg/s}\,.$$

   - **(b) Assuming the star has constant luminosity and starts as 75% ^{1}H, in how many years will it run out of fuel?**

This is just the mass divided by the mass consumption rate, or

$$t = \frac{M}{dM/dt} = \frac{0.75(0.080)\left(1.989\times10^{30}\text{ kg}\right)}{7.535\times10^{7}\text{ kg/s}} = \frac{1.584\times10^{21}\text{ s}}{3.156\times10^{7}\text{ s/yr}} = 5.02\times10^{13}\text{ yr}\,.$$

This fits rather well with our estimate of 10^{14} yr from class.

2. **[20] Black holes evaporate according to formulas provided in the lectures. Find each of the following for a black hole of mass (i) $10\,M_\odot$ and (ii) $10^{11}\,M_\odot$:**
   - **(a) The Schwarzschild radius in m.**

The Schwarzschild radius is just $R_S = 2GM/c^2$, so we have

$$R_* = \frac{2\left(6.674\times10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}\right)\left(10\right)\left(1.989\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^{8}\ \text{m/s}\right)^2} = 2.954\times10^{4}\ \text{m},$$

$$R_g = \frac{2\left(6.674\times10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}\right)\left(10^{11}\right)\left(1.989\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^{8}\ \text{m/s}\right)^2} = 2.954\times10^{14}\ \text{m}.$$

(b) The Hawking temperature in K.

The Hawking temperature is given by $k_B T = \hbar c/\left(4\pi R_S\right)$, so we have

$$T_* = \frac{\hbar c}{4\pi k_B R_s} = \frac{1.973\times10^{-7}\ \text{eV}\cdot\text{m}}{4\pi\left(8.617\times10^{-5}\ \text{eV/K}\right)\left(2.954\times10^{4}\ \text{m}\right)} = 6.169\times10^{-9}\ \text{K},$$

$$T_g = \frac{\hbar c}{4\pi k_B R_s} = \frac{1.973\times10^{-7}\ \text{eV}\cdot\text{m}}{4\pi\left(8.617\times10^{-5}\ \text{eV/K}\right)\left(2.954\times10^{14}\ \text{m}\right)} = 6.169\times10^{-19}\ \text{K}.$$

These are very cold.

(c) The luminosity in W.

We simply use the formula we would normally use for the luminosity of a star, namely

$$\begin{aligned} L_* &= 4\pi\sigma R_*^2 T_*^4 = 4\pi\left(5.670\times10^{-8}\ \text{W/m}^2/\text{K}^4\right)\left(2.954\times10^{4}\ \text{m}\right)^2\left(6.169\times10^{-9}\ \text{K}\right)^4 \\ &= 9.005\times10^{-31}\ \text{W}, \\ L_g &= 4\pi\sigma R_*^2 T_*^4 = 4\pi\left(5.670\times10^{-8}\ \text{W/m}^2/\text{K}^4\right)\left(2.954\times10^{16}\ \text{m}\right)^2\left(6.169\times10^{-19}\ \text{K}\right)^4 \\ &= 9.005\times10^{-51}\ \text{W}. \end{aligned}$$

(d) The approximate time in yr for the black hole's energy *Mc*² to be completely evaporated.

We simply divide the starting energy by the rate of energy loss to yield

$$t_* = \frac{M_* c^2}{L_*} = \frac{10\left(1.989\times10^{30}\ \text{kg}\right)\left(2.998\times10^{8}\ \text{m/s}\right)^2}{9.005\times10^{-31}\ \text{W}} = \frac{1.985\times10^{78}\ \text{s}}{3.156\times10^{7}\ \text{s/yr}} = 6.29\times10^{70}\ \text{yr},$$

$$t_g = \frac{M_g c^2}{L_g} = \frac{10^{11}\left(1.989\times10^{30}\ \text{kg}\right)\left(2.998\times10^{8}\ \text{m/s}\right)^2}{9.005\times10^{-51}\ \text{W}} = \frac{1.985\times10^{108}\ \text{s}}{3.156\times10^{7}\ \text{s/yr}} = 6.29\times10^{100}\ \text{yr},$$

Graduate Problems: There are no graduate problems for this homework.