Physics 712
Solutions to Chapter 11 Problems

3. A pion (mass m_π) at rest decays to a muon (mass m_μ) and a neutrino (mass 0). Find the energies of the two final particles.

We first define the momenta in an obvious way, then we write conservation of four-momentum as

$$p_\pi = p_\mu + p_\nu$$

If we solve for, say, the muon momentum, we have $p_\mu = p_\pi - p_\nu$. Dotting this into itself, we have

$$p_\mu \cdot p_\mu = p_\pi \cdot p_\pi + p_\nu \cdot p_\nu - 2 p_\pi \cdot p_\nu$$

We replace all the dot products of the momenta with themselves by $p \cdot p = m^2 c^2$, and we have

$$m_\mu^2 c^2 = m_\pi^2 c^2 + 0 - 2 p_\pi \cdot p_\nu$$

The initial pion has momentum $p_\pi = \left(m_\pi c, 0, 0, 0\right)$, and we write the neutrino momentum as $p_\nu = \left(E_\nu / c, \mathbf{p}_\nu\right)$. The dot product is then $p_\pi \cdot p_\nu = m_\pi E_\nu$, and we have

$$m_\mu^2 c^2 = m_\pi^2 c^2 + 0 - 2 m_\pi E_\nu ,$$
$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2 m_\pi} c^2$$

To get the muon energy, the easiest way is to use conservation of energy:

$$E_\mu = E_\pi - E_\nu = m_\pi c^2 - \frac{m_\pi^2 - m_\mu^2}{2 m_\pi} c^2 = \frac{2 m_\pi^2 - m_\pi^2 + m_\mu^2}{2 m_\pi} c^2 = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi} c^2 .$$

4. A particle of mass *m* and charge *q* is in the presence of constant electric and magnetic fields $\mathbf{E} = E\hat{\mathbf{x}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$.
(a) Write out explicitly all four components of the equation for $\dot{U}^\mu$, where dot stands for $d/d\tau$. Find an equation for $\ddot{U}^1$.

The electromagnetic field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E/c & 0 & 0 \\ E/c & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{so} \quad F^\mu{}_\nu = \begin{pmatrix} 0 & E/c & 0 & 0 \\ E/c & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where we lowered the index by changing the sign of the last three columns.

We now need to solve the equations

$$m\dot{U}^{\mu} = qF^{\mu}{}_{\nu}U^{\nu}\,, \quad \text{or} \quad m\begin{pmatrix}\dot{U}^0\\ \dot{U}^1\\ \dot{U}^2\\ \dot{U}^3\end{pmatrix} = q\begin{pmatrix}0 & E/c & 0 & 0\\ E/c & 0 & B & 0\\ 0 & -B & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}U^0\\ U^1\\ U^2\\ U^3\end{pmatrix} = \begin{pmatrix}qEU^1/c\\ qEU^0/c + qBU^2\\ -qBU^1\\ 0\end{pmatrix}$$

This breaks into four separate equations:

$$\dot{U}^0 = \frac{qE}{mc}U^1\,, \quad m\dot{U}^1 = \frac{qE}{mc}U^0 + \frac{qB}{m}U^2\,, \quad \dot{U}^2 = -\frac{qB}{m}U^1\,, \quad \dot{U}^3 = 0\,.$$

The last equation is always trivial to solve.

To get a second order differential equation for U^1, take another time derivative of the second equation and substitute the first and third equation.

$$\ddot{U}^1 = \frac{qE}{mc}\dot{U}^0 + \frac{qB}{m}\dot{U}^2 = \frac{qE^2}{m^2c^2}U^1 - \frac{qB^2}{m^2}U^1 = \frac{q^2}{m^2c^2}\left(E^2 - c^2B^2\right)U^1$$

(b) What is the general solution for $U^1(\tau)$ part (b) if $E < cB$? Argue that it will exhibit periodic behavior (in τ), and find the period.

If $E < cB$, then we define

$$\omega = \frac{q}{mc}\sqrt{B^2c^2 - E^2}$$

Then our equation is $\ddot{U}^1 = -\omega^2 U^1$, whose general solution is

$$U^1 = a\cos(\omega\tau) + b\sin(\omega\tau)\,.$$

This will exhibit periodic behavior with a period of

$$T = \frac{2\pi}{\omega} = \frac{2\pi mc}{q\sqrt{E^2 - c^2B^2}}\,.$$

(c) Repeat part (b) if $E > cB$. Will it be periodic in this case?

If $E > cB$, then define

$$\alpha = \frac{q}{mc}\sqrt{E^2 - B^2c^2}$$

Then our equation is $\ddot{U}^1 = \alpha^2 U^1$, whose general solution is

$$U^1 = a\cosh(\alpha\tau) + b\sinh(\alpha\tau)\,.$$

This does not exhibit periodic behavior.