Physics 712
Solutions to Chapter 11 Problems

5. Consider a line of charge with linear charge density λ arranged, in a primed frame, along the *y′*-axis at rest. Write the electric field at all points in Cartesian coordinates in the primed frame. Now, consider a line of charge with the same linear charge density, parallel to the *y*-axis, but this time moving in the +*x* direction at velocity *v*. Find the electric and magnetic fields everywhere in the unprimed frame.

For a line of charge along the y'-axis, we can draw a cylinder of radius r' and length L around the linear charge density. The charge enclosed will be λL. Symmetry argues that the electric field will point directly out of the cylinder on the lateral surface, and will depend only on the distance away, so that $\mathbf{E}' = E'(r)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector pointing away from the y'-axis. We then use Gauss's Law to conclude that the electric field everywhere is

$$\frac{\lambda L}{\varepsilon_0} = \int_S \mathbf{E}' \cdot \hat{\mathbf{n}}\, da = 2\pi r' L E'(r'), \quad \text{so} \quad E'(r') = \frac{\lambda}{2\pi\varepsilon_0 r'} .$$

We therefore have

$$\mathbf{E} = E'(r)\hat{\mathbf{r}} = \frac{\lambda\hat{\mathbf{r}}}{2\pi\varepsilon_0 r'} = \frac{\lambda\mathbf{r}'}{2\pi\varepsilon_0 r'^2} = \frac{\lambda\left(x'\hat{\mathbf{x}} + z'\hat{\mathbf{z}}\right)}{2\pi\varepsilon_0\left(x'^2 + z'^2\right)},$$

where we recall that in this context r' is the distance of the point from the y'-axis. Of course, in the primed frame, there is no magnetic field at all.

To solve the "harder" problem, we now simply perform a Lorentz boost by speed $-v$ in the x-direction. There is one (apparent) subtlety here – are we sure the linear charge density λ is the same in both frames? We know that charge is Lorentz-invariant, and a boost in the x-direction does not affect distances in the y-direction, and since linear charge density is the charge per unit length (in the y-direction), the linear charge density should be unchanged.

The Lorentz transformations for the fields for this Lorentz boost will be

$$\mathbf{E}_{\parallel} = \mathbf{E}'_{\parallel} = \frac{\lambda x'\hat{\mathbf{x}}}{2\pi\varepsilon_0\left(x'^2 + z'^2\right)}, \quad \mathbf{E}_{\perp} = \gamma\left(\mathbf{E}'_{\perp} - \mathbf{v}\times\mathbf{B}'\right) = \frac{\gamma\lambda z'\hat{\mathbf{z}}}{2\pi\varepsilon_0\left(x'^2 + z'^2\right)},$$

$$\mathbf{B}_{\parallel} = \mathbf{B}'_{\parallel} = 0, \quad \mathbf{B}_{\perp} = \gamma\left(\mathbf{B}'_{\perp} + \mathbf{v}\times\mathbf{E}'/c^2\right) = \frac{\gamma v\hat{\mathbf{x}}\times\lambda z'\hat{\mathbf{z}}}{2\pi\varepsilon_0 c^2\left(x'^2 + z'^2\right)} = \frac{-\gamma\mu_0\lambda z'\hat{\mathbf{y}}}{2\pi\left(x'^2 + z'^2\right)} .$$

The coordinates are related by

$$t' = \gamma\left(t - vx/c^2\right), \quad x' = \gamma\left(x - vt\right), \quad y' = y, \quad z' = z.$$

Substituting this into the previous expressions, we have

$$\mathbf{E} = \frac{\lambda\gamma\left[\left(x - vt\right)\hat{\mathbf{x}} + z\hat{\mathbf{z}}\right]}{2\pi\varepsilon_0\left[\gamma^2\left(x - vt\right)^2 + z^2\right]}, \quad \mathbf{B} = \frac{-\gamma\mu_0\lambda z\hat{\mathbf{y}}}{2\pi\left[\gamma^2\left(x - vt\right)^2 + z^2\right]}, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} .$$