Physics 712
Chapter 1 Problems

6. [5] Estimate the capacitance of a sphere of radius *R* using the trial potential function $\Psi(\mathbf{x}) = e^{-\lambda(r-R)/2}$, and compare to the exact value from lecture.

An upper bound on the capacitance is given by

$$C \le S(\lambda) = \varepsilon_0 \int_V (\nabla\Psi)^2 d^3\mathbf{x} = \varepsilon_0 \int_a^\infty (\partial\Psi/\partial r)^2 4\pi r^2 dr = 4\pi\varepsilon_0 \tfrac{1}{4}\lambda^2 \int_a^\infty e^{-\lambda(r-R)} r^2 dr$$

$$= \pi\varepsilon_0 \lambda^2 \int_0^\infty e^{-\lambda w}(w+R)^2 dw = \pi\varepsilon_0\lambda^2\left(2\lambda^{-3} + 2R\lambda^{-2} + R^2\lambda^{-1}\right) = \pi\varepsilon_0\left(2\lambda^{-1} + 2R + R^2\lambda\right)$$

We now minimize this with respect to the variational parameter λ, which yields

$$0 = -2\lambda^{-2} + R^2 , \quad \text{so} \quad \lambda = \sqrt{2}/R .$$

Substituting this back into the formula, we have

$$C \le S\left(\sqrt{2}/R\right) = \pi\varepsilon_0\left(\sqrt{2}R + 2R + R^2\sqrt{2}\right) = \left(2 + 2\sqrt{2}\right)\pi\varepsilon_0 R = 4.828\pi\varepsilon_0 R .$$

The actual value, as calculated in a sample problem, is $C = 4\pi\varepsilon_0 R$, so our answer is 20% high.

7. **[10] Consider a square of side *a* with *V* = 0 on three sides and *V* = 1 on the surface *y* = *a* in two dimensions. Our goal is to compute the potential $\Phi\left(\frac{1}{4}a,\frac{1}{2}a\right)$ using the relaxation method. To do so, you can download some helpful spreadsheets at**

http://users.ecarlson.wfu/eandm/relax.xlsx

   (a) We will first work in a low-resolution matrix with grid spacing $\frac{1}{4}a$. Check that the formula in square B2 is correct for computing using Φ_{ij}^{+}, then copy it and paste the formula into the rest of the interior of the spreadsheet (by selecting Paste, f_x from the pulldown menu). Then press F9 repeatedly to force recalculation until it converges.

The formula for Φ_{ij}^{+} is supposed to be the average of the four adjacent cells, so indeed, the formula B2 = (B1+A2+C2+B3)/4 is indeed correct.

   (b) Now switch to the medium-res tab at the bottom with grid spacing $\frac{1}{8}a$. Redo the calculation. Then switch to the high-res tab and redo it with spacing $\frac{1}{16}a$.

It is probably worth noting that, because the grid is larger, it takes many more iterations at higher resolution.

   (c) Now, redo the formula so we are instead using $\Phi_{ij}=\frac{4}{5}\Phi_{ij}^{+}+\frac{1}{5}\Phi_{ij}^{\times}$. Redo the calculation in each case. Record all the values for $\Phi\left(\frac{1}{4}a,\frac{1}{2}a\right)$ in a table (it should have six values in it).

Since each of these is already an average with a factor of 4 in the denominator, the first term ends with a factor of 5 and the remaining terms with a factor of 20, so the formula becomes B2 = (B1+A2+C2+B3)/5 + (A1+A3+C1+C3)/20. You then cut and paste this formula everywhere.

	low	med	high
Φ_{ij}^{+}	0.1875	0.183823529	0.18251617
$\Phi_{ij}=\frac{4}{5}\Phi_{ij}^{+}+\frac{1}{5}\Phi_{ij}^{\times}$	0.181818182	0.182015095	0.182027585

   (d) Comment on which technique you think is most accurate. Based on all your computations, how accurate do you think your final answer is? (about how many digits?)

Since the weighted average (the second row) is expected to be more accurate, we focus on that row. Note that the low and medium resolution agree to about three digits, and the medium to high resolution agree to about five digits. This suggests that if we went still higher, the high resolution would probably agree to about seven digits, so we expect the correct answer is around 0.1820276 or so.