Physics 712
Chapter 2 Problems

3. [15] Consider a square of side *a* with Φ = 0 on three sides and Φ = *V* on the surface *y* = *a* in two dimensions. Our goal is to compute the potential everywhere, and particularly $\Phi\left(\tfrac{1}{4}a,\tfrac{1}{2}a\right)$. Write the potential in the form $\Phi(x,y)=\sum_{n=1}^{\infty}A_n(y)\sin(n\pi x/a)$. What is the form of the functions $A_n(y)$? By matching appropriate boundary conditions, determine any unknown coefficients, and find $\Phi(x,y)$ as an infinite sum. Sum it numerically to find $\Phi\left(\tfrac{1}{4}a,\tfrac{1}{2}a\right)$. Compare your results with the results of problem 1.7.

We want the Laplacian to vanish, so this implies

$$0=\nabla^2\Phi(x,y)=\sum_{n=1}^{\infty}\left[\frac{\partial^2}{\partial y^2}A_n(y)-\frac{\pi^2n^2}{a^2}A_n(y)\right]\sin\left(\frac{\pi nx}{a}\right).$$

Since the sine functions are an independent set of functions, the only way this can vanish is if the expression in square brackets vanishes. We therefore have

$$\frac{d^2}{dy^2}A_n(y)=\frac{\pi^2n^2}{a^2}A_n(y).$$

This has general solution

$$A_n(y)=\alpha_n e^{\pi ny/a}+\beta_n e^{-\pi ny/a}\ .$$

However, this function must also vanish at *y* = 0, so this implies $\beta_n=-\alpha_n$, and our function is

$$\Phi(x,y)=\sum_{n=1}^{\infty}2\alpha_n\sin\left(\frac{\pi nx}{a}\right)\sinh\left(\frac{\pi ny}{a}\right)$$

We now start working on the constants α_n. We note that if we set *y* = *a*, we must have

$$V=\sum_{n=1}^{\infty}2\alpha_n\sin\left(\frac{\pi nx}{a}\right)\sinh(\pi n)$$

If we multiply both sides of this equation by $\sin(\pi mx/a)$ and integrate over *x*, we can use the fact that the functions $\sin(\pi mx/a)$ are orthogonal to find

$$\int_0^a V\sin\left(\frac{\pi mx}{a}\right)dx=\sum_{n=1}^{\infty}2\alpha_n\int_0^a\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi mx}{a}\right)dx\sinh(\pi n),$$

$$-\frac{Va}{\pi m}\cos\left(\frac{\pi mx}{a}\right)\bigg|_{m=0}^{a}=\sum_{n=1}^{\infty}2\alpha_n\tfrac{1}{2}a\delta_{nm}\sinh(\pi n),$$

$$\frac{V}{\pi m}\left[1-(-1)^m\right]=\alpha_m\sinh(\pi m)$$

The expression in square brackets vanishes for *m* even and is 2 for *m* odd. Substituting this back into our expression for the potential, we have

$$\Phi(x,y)=\sum_{n\text{ odd}}^{\infty}\frac{4V}{\pi n\sinh(\pi n)}\sin\left(\frac{\pi nx}{a}\right)\sinh\left(\frac{\pi ny}{a}\right)$$

We have been asked to compute $\Phi\left(\frac{1}{4}a,\frac{1}{2}a\right)$, which is therefore

$$\Phi\left(\tfrac{1}{4}a,\tfrac{1}{2}a\right)=\sum_{n\text{ odd}}^{\infty}\frac{4V}{\pi n\sinh(\pi n)}\sin\left(\tfrac{1}{4}\pi n\right)\sinh\left(\tfrac{1}{2}\pi n\right)=V\sum_{n\text{ odd}}^{\infty}\frac{2\sin\left(\frac{1}{4}\pi n\right)}{\pi n\cosh\left(\frac{1}{2}\pi n\right)},$$

where at the last step we used the double angle formula $\sinh(2\theta)=2\sinh\theta\cosh\theta$ to simplify a bit. We now let Maple do the sum numerically for us:

```
> add(evalf(sin(Pi*(2*n-1)/4)/cosh(Pi*(n-1/2))/Pi/(n-1/2)),
  n=1..6);
```

We find $\Phi\left(\frac{1}{4}a,\frac{1}{2}a\right)=0.1820283319V$, adding just six terms. In problem 1.7, our best estimate was $\Phi\left(\frac{1}{4}a,\frac{1}{2}a\right)=0.182027585V$, which is correct to about six digits, so I was a bit optimistic there when I claimed seven digits.