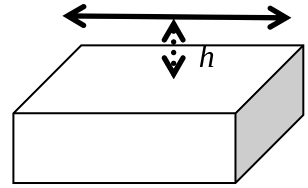
Physics 712
Chapter 4 Problems

3. [15] An infinite line charge with charge per unit length $\lambda = 0$ parallel to the *x*-axis a distance *h* above a semi-infinite dielectric with dielectric constant ε. Find the force per unit length on the line charge. Find the bound surface charge density σ_b on the surface of the dielectric.



We can think of a line of charge as if it were a series of point charges of length *dx* each of which has charge λdx. When calculating the resulting electric field above the plane, we have to add image charges below the plane. It is clear that this will correspond to just a line of charges at $z = -h$ of magnitude λ'. Similarly, when calculating the electric field within the dielectric, it will look like it is coming from a line charge λ'' located above the plane at $z = h$. The magnitude of these fictitious line charges will be

$$\lambda' = \frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon}\lambda, \quad \lambda'' = \frac{2\varepsilon}{\varepsilon + \varepsilon_0}\lambda .$$

The electric field from a line charge in vacuum was calculated long ago. It is given, for the line charge itself, by

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda\hat{\boldsymbol{\rho}}}{2\pi\varepsilon_0\rho} = \frac{\lambda\boldsymbol{\rho}}{2\pi\varepsilon_0\rho^2},$$

where $\boldsymbol{\rho}$ is a vector pointing from the line to an arbitrary point. It will be given by

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda\left[y\hat{\mathbf{y}} + (z-h)\hat{\mathbf{z}}\right]}{2\pi\varepsilon_0\left[y^2 + (z-h)^2\right]}.$$

We now add to this the contribution from the image charges, so that for $z > 0$, the electric field will be

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda\left[y\hat{\mathbf{y}} + (z-h)\hat{\mathbf{z}}\right]}{2\pi\varepsilon_0\left[y^2 + (z-h)^2\right]} - \frac{\lambda(\varepsilon - \varepsilon_0)\left[y\hat{\mathbf{y}} + (z+h)\hat{\mathbf{z}}\right]}{2\pi\varepsilon_0(\varepsilon + \varepsilon_0)\left[y^2 + (z+h)^2\right]} \quad \text{for} \quad z > 0.$$

Below the plane, the main change is we need to remember to divide by ε, since we are in the medium. We therefore have

$$\mathbf{E}(\mathbf{x}) = \frac{2\varepsilon\lambda\left[y\hat{\mathbf{y}} + (z-h)\hat{\mathbf{z}}\right]}{2\pi\varepsilon(\varepsilon + \varepsilon_0)\left[y^2 + (z-h)^2\right]} = \frac{\lambda\left[y\hat{\mathbf{y}} + (z-h)\hat{\mathbf{z}}\right]}{\pi(\varepsilon + \varepsilon_0)\left[y^2 + (z-h)^2\right]} \quad \text{for} \quad z < 0.$$

To find the force, we realize that the force is due to the electric field from the image line charge on the actual charge. The force on a charge is $\mathbf{F} = q\mathbf{E}$, so the force per unit length on the line charge would be $\mathbf{F}/L = q\mathbf{E}/L = \lambda\mathbf{E}$, so

$$\frac{\mathbf{F}}{L} = \lambda\mathbf{E} = -\frac{\lambda^2(\varepsilon-\varepsilon_0)\left[0\hat{\mathbf{y}}+(h+h)\hat{\mathbf{z}}\right]}{2\pi\varepsilon_0(\varepsilon+\varepsilon_0)\left[0^2+(h+h)^2\right]} = -\frac{\lambda^2(\varepsilon-\varepsilon_0)\hat{\mathbf{z}}}{4\pi\varepsilon_0(\varepsilon+\varepsilon_0)h}.$$

To find the bound surface charge density, we simply find $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{z}}$ at $z = 0$, to give

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{z}} = (\mathbf{D}-\varepsilon_0\mathbf{E})\cdot\hat{\mathbf{z}} = (\varepsilon-\varepsilon_0)\mathbf{E}\cdot\hat{\mathbf{z}} = \frac{\lambda(\varepsilon-\varepsilon_0)\left[y\hat{\mathbf{y}}+(0-h)\hat{\mathbf{z}}\right]\cdot\hat{\mathbf{z}}}{\pi(\varepsilon+\varepsilon_0)\left[y^2+(0-h)^2\right]} = \frac{\lambda h(\varepsilon_0-\varepsilon)}{\pi(\varepsilon+\varepsilon_0)(y^2+h^2)}.$$

Interestingly, if you integrate this over y, you will get the linear charge density λ', and this is not a coincidence.

4. [15] A dielectric sphere with dielectric constant ε of radius a lies at the origin in a background potential (in the absence of the sphere) of the form $\Phi(\mathbf{x}) = \lambda xy$.

(a) Write the background potential in terms of spherical harmonics times powers of r.

We first note that the background potential, in spherical coordinates, is

$$\Phi(\mathbf{x}) = \lambda r^2\sin^2\theta\sin\phi\cos\phi = \tfrac{1}{2}\lambda r^2\sin^2\theta\sin(2\phi).$$

Comparing this to the spherical harmonics $Y_{2,\pm2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi}$, it is not hard to see that

$$\lambda xy = \tfrac{1}{2}\lambda r^2\sin^2\theta\sin(2\phi) = \frac{\lambda}{4i}r^2\sin^2\theta\left(e^{2i\phi}-e^{-2i\phi}\right) = \frac{4\lambda}{4i}\sqrt{\frac{2\pi}{15}}r^2\left[Y_{2,2}(\theta,\phi)-Y_{2,-2}(\theta,\phi)\right]$$

(b) Write a reasonable conjecture for the form of the potential in the regions $r < a$ and $r > a$. Your conjecture should automatically satisfy $\nabla^2\Phi = 0$ within each of these regions. It may contain unknown constants.

We recall that $\nabla^2\Phi = 0$ for an arbitrary potential of the form

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\alpha_{lm}r^l + \beta_{lm}r^{-l-1}\right)Y_{lm}(\theta,\phi)$$

Our conjecture is now:

$$\Phi_{\text{out}}(r,\theta,\phi) = \lambda i\sqrt{\frac{2\pi}{15}}\left(r^2+\beta r^{-3}\right)\left[Y_{2,-2}(\theta,\phi)-Y_{2,2}(\theta,\phi)\right],$$

$$\Phi_{\text{in}}(r,\theta,\phi) = \lambda i\sqrt{\frac{2\pi}{15}}\left(\alpha r^2\right)\left[Y_{2,-2}(\theta,\phi)-Y_{2,2}(\theta,\phi)\right],$$

This automatically satisfies Laplace's equation. Furthermore, it has the correct asymptotic form at $r = \infty$ and is well-behaved at $r = 0$.

(c) By matching suitable boundary conditions, determine the value of any unknown constants.

Since we are working with the potential, it makes sense to match the potential at the boundary $r = a$. We therefore have

$$a^2 + \beta a^{-3} = \alpha a^2 , \quad \text{or} \quad \alpha = 1 + \beta a^{-5} .$$

Matching this equation will automatically insure that $\mathbf{E}_{\parallel}$ is continuous as well.

It remains to make sure that $\mathbf{D}_{\perp}$. To make this match, we need

$$\varepsilon \mathbf{E}_{\perp\text{in}} = \varepsilon_0 \mathbf{E}_{\perp\text{out}} ,$$

$$\varepsilon \frac{\partial}{\partial r} \Phi_{\text{in}}(r)\Big|_{r=a} = \varepsilon_0 \frac{\partial}{\partial r} \Phi_{\text{out}}(r)\Big|_{r=a} ,$$

$$\varepsilon \lambda i \sqrt{\frac{2\pi}{15}} (2\alpha a) \left[Y_{2,-2}(\theta,\phi) - Y_{2,2}(\theta,\phi) \right] = \varepsilon_0 \lambda i \sqrt{\frac{2\pi}{15}} \left(2a - 3\beta a^{-4} \right) \left[Y_{2,-2}(\theta,\phi) - Y_{2,2}(\theta,\phi) \right] ,$$

$$2\alpha\varepsilon = \left(2 - 3\beta a^{-5} \right) \varepsilon_0 .$$

If we substitute our previous equation into this one, we find

$$2\varepsilon \left(1 + \beta a^{-5} \right) = \left(2 - 3\beta a^{-5} \right) \varepsilon_0 ,$$

$$2(\varepsilon - \varepsilon_0) = -(3\varepsilon_0 + 2\varepsilon) \beta a^{-5} ,$$

$$\beta = -\frac{2(\varepsilon - \varepsilon_0)}{3\varepsilon_0 + 2\varepsilon} a^5 .$$

We also find

$$\alpha = 1 + \beta a^{-5} = 1 - \frac{2(\varepsilon - \varepsilon_0)}{3\varepsilon_0 + 2\varepsilon} = \frac{5\varepsilon_0}{3\varepsilon_0 + 2\varepsilon} .$$

Substituting these two expressions back in, and getting rid of the spherical harmonics, we have

$$\Phi_{\text{out}}(r,\theta,\phi) = \left(\lambda r^2 - \frac{2\varepsilon - 2\varepsilon_0}{3\varepsilon_0 + 2\varepsilon} \cdot \frac{\lambda a^5}{r^3} \right) \sin^2\theta \cos\phi \sin\phi ,$$

$$\Phi_{\text{in}}(r,\theta,\phi) = \frac{5\varepsilon_0}{3\varepsilon_0 + 2\varepsilon} \lambda r^2 \sin^2\theta \cos\phi \sin\phi .$$