Physics 712
Solution to Problems 4.5 and 5.1

5. [5] For problem 4.1, find the total energy if the cylinder has length *L*. For problem 4.2, find the total energy. In each case, show that the answer is equivalent to $W = \tfrac{1}{2} Q \Delta\Phi$.

The energy is given for problem 4.1 by

$$W = \tfrac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\, d^3\mathbf{x} = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^L dz \left[\int_a^b \varepsilon \left(\frac{\lambda}{2\pi\varepsilon\rho}\right)^2 \rho\, d\rho + \int_b^c \varepsilon_0 \left(\frac{\lambda}{2\pi\varepsilon_0\rho}\right)^2 \rho\, d\rho\right]$$

$$= \frac{2\pi\lambda^2 L}{8\pi^2}\left(\frac{1}{\varepsilon}\ln\rho\Big|_a^b + \frac{1}{\varepsilon_0}\ln\rho\Big|_b^c\right) = \frac{\lambda^2 L}{4\pi}\left[\frac{1}{\varepsilon}\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_0}\ln\left(\frac{c}{b}\right)\right] = \tfrac{1}{2}\lambda L \Delta\Phi = \tfrac{1}{2} Q\Delta\Phi\,,$$

where at the last step, we interpreted $\lambda L = Q$ as the total charge. For problem 4.2, we have

$$W = \tfrac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\, d^3\mathbf{x} = \frac{1}{2}\int_0^{2\pi} d\phi \int_a^b \left[\frac{Q}{\pi\left(3\varepsilon_0 + \varepsilon\right) r^2}\right]^2 r^2 dr \left(\int_0^{\frac{1}{3}\pi} \varepsilon \sin\theta\, d\theta + \int_{\frac{1}{3}\pi}^{\pi} \varepsilon_0 \sin\theta\, d\theta\right)$$

$$= \frac{2\pi Q^2}{2\pi^2\left(3\varepsilon_0+\varepsilon\right)^2} \frac{-1}{r}\Bigg|_a^b \left(-\varepsilon\cos\theta\Big|_0^{\frac{1}{3}\pi} - \varepsilon_0\cos\theta\Big|_{\frac{1}{3}\pi}^{\pi}\right) = \frac{Q^2}{\pi\left(3\varepsilon_0+\varepsilon\right)^2}\left(\frac{1}{a}-\frac{1}{b}\right)\left(\tfrac{1}{2}\varepsilon + \tfrac{3}{2}\varepsilon_0\right)$$

$$= \frac{Q^2\left(b-a\right)}{2\pi ab\left(3\varepsilon_0+\varepsilon\right)} = \tfrac{1}{2} Q\Delta\Phi\,.$$

1. **[10] We are trying to trap a charged particle of mass $q > 0$ and mass m by using a combination of magnetic and electric fields given by $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{E} = A\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\right)$.**

   (a) [1] Obviously, $\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0$. Check that it also satisfies $\nabla \cdot \mathbf{E} = \nabla \times \mathbf{E} = 0$.

   We simply see that $\nabla \cdot \mathbf{E} = A + A - 2A = 0$, and all the terms in $\nabla \times \mathbf{E}$ vanish.

   (b) [6] Assume the particle has motion given by $x = R\cos(\omega t)$, $y = R\sin(\omega t)$. Find an equation for ω in terms of A and B.

   The velocity and acceleration can be found by simply taking derivatives:

$$\mathbf{v} = \dot{\mathbf{x}} = R\omega\left[-\sin(\omega t)\hat{\mathbf{x}} + \cos(\omega t)\right], \quad \mathbf{a} = \dot{\mathbf{v}} = R\omega^2\left[-\cos(\omega t)\hat{\mathbf{x}} - \sin(\omega t)\right]$$

We therefore have

$$m\mathbf{a} = \mathbf{F} = \left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) = qRA\left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right] + qBR\omega\left[-\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right]\times\hat{\mathbf{z}},$$

$$-mR\omega^2\left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right] = qRA\left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right] + qBR\omega\left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right],$$

$$-mR\omega^2 = qRA + qBR\omega,$$

$$m\omega^2 + qB\omega + qA = 0.$$

We then solve this using the quadratic equation, so

$$\omega = \frac{-qB \pm \sqrt{q^2B^2 - 4mqA}}{2m}.$$

Until I solved this problem myself, I didn't even realize there were two solutions to this equation.

   (c) [3] Argue that there is a maximum value of A for which circular motion is possible. Also argue that for $A > 0$, the particle will not "wander off" in the z-direction.

   The solution only makes sense if the discriminant is positive, so we must have $q^2B^2 \geq 4mqA$, or $A \leq qB^2/(4m)$. Although we have not discusses motion in the z-direction, it is pretty easy to see that the magnetic field has no influence on it, so the only vertical force is $F_z = E_z q = -2Azq$. Such a linear restoring force will result in simple harmonic motion in the z-direction, so it is stable against motion in the z-direction.