Physics 712
Chapter 7 Solutions

1. In Richard Williams' lab, a laser can (briefly) produce 50 GW of power and be focused on a region of size 1 μm². How large are the maximum electric and magnetic fields?

The intensity of the beam is the power over the area, or

$$I = \frac{5.0\times10^{10}\ \text{W}}{\left(1.0\times10^{-6}\ \text{m}\right)^2} = 5.0\times10^{22}\ \text{W/m}^2 .$$

We then equate this to the magnitude of the Poynting vector. We have

$$I = \left|\langle\mathbf{S}\rangle\right| = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{E}_0\cdot\mathbf{E}_0^*,$$

$$\mathbf{E}_0\cdot\mathbf{E}_0^* = 2\sqrt{\frac{\mu_0}{\varepsilon_0}}I = \frac{2\mu_0}{\sqrt{\mu_0\varepsilon_0}}I = 2c\mu_0 I = 2\left(2.998\times10^8\ \text{m/s}\right)\left(4\pi\times10^{-7}\ \text{N/A}^2\right)\left(5.0\times10^{22}\ \text{W/m}^2\right)$$

$$= 3.76\times10^{25}\ \text{N}^2\cdot\text{A}^{-2}\cdot\text{s}^{-2} = 3.76\times10^{25}\ \text{N}^2\cdot\text{C}^{-2},$$

$$\left|\mathbf{E}_0\right| = \sqrt{3.76\times10^{25}\ \text{N}^2\cdot\text{C}^{-2}} = 6.14\times10^{12}\ \text{N/C}.$$

The magnetic fields are given by

$$\mathbf{B}_0 = \sqrt{\mu_0\varepsilon_0}\,\hat{\mathbf{k}}\times\mathbf{E}_0 = \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{E}_0,$$

$$\left|\mathbf{B}_0\right| = \frac{1}{c}\left|\mathbf{E}_0\right| = \frac{6.14\times10^{12}\ \text{N/C}}{2.998\times10^8\ \text{m/s}} = 20{,}500\ \text{T}.$$

By comparison, a very strong static electric field would be about 10^8 N/m and a big magnetic field would be 100 T.

2. Suppose a perfect polarizer extracts from a pure wave in the *z*-direction just the polarization $\boldsymbol{\varepsilon}_x = \hat{\mathbf{x}}$, $\boldsymbol{\varepsilon}_y = \hat{\mathbf{y}}$, $\boldsymbol{\varepsilon}_/ = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + \hat{\mathbf{y}}\right)$, or $\boldsymbol{\varepsilon}_\backslash = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} - \hat{\mathbf{y}}\right)$. In each case, write the resulting intensity in terms of just the Stokes parameters. Find a relationship between the four intensities $I_x, I_y, I_/, I_\backslash$.

The most general wave we can have is of the form $\mathbf{E}_0 = E_1\hat{\mathbf{x}} + E_2\hat{\mathbf{y}}$. It is obvious that if we use the first two cases, the resulting electric fields will be just $\mathbf{E}_0 = E_1\hat{\mathbf{x}}$ or $\mathbf{E}_0 = E_2\hat{\mathbf{y}}$, and the resulting intensities will be

$$I_x = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} E_1 E_1^*, \quad I_y = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} E_2 E_2^*$$

On the other hand, we can also write the same wave in the form

$$\mathbf{E}_0 = E_1\hat{\mathbf{x}} + E_2\hat{\mathbf{y}} = \tfrac{1}{2}\left(E_1 + E_2\right)\left(\hat{\mathbf{x}} + \hat{\mathbf{y}}\right) + \tfrac{1}{2}\left(E_1 - E_2\right)\left(\hat{\mathbf{x}} - \hat{\mathbf{y}}\right) = \tfrac{1}{\sqrt{2}}\left(E_1 + E_2\right)\boldsymbol{\varepsilon}_/ + \tfrac{1}{\sqrt{2}}\left(E_1 - E_2\right)\boldsymbol{\varepsilon}_\backslash .$$

If we extract out just one of these two polarizations, it is evidence that we have

$$\begin{aligned} I_/ &= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{2}\left(E_1 + E_2\right)\left(E_1^* + E_2^*\right) = \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left(E_1E_1^* + E_2E_2^* + E_1E_2^* + E_2E_1^*\right) \\ &= \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left[E_1E_1^* + E_2E_2^* + 2\,\mathrm{Re}\left(E_1^*E_2\right)\right], \\ I_\backslash &= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{2}\left(E_1 - E_2\right)\left(E_1^* - E_2^*\right) = \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left(E_1E_1^* + E_2E_2^* - E_1E_2^* - E_2E_1^*\right) \\ &= \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left[E_1E_1^* + E_2E_2^* - 2\,\mathrm{Re}\left(E_1^*E_2\right)\right]. \end{aligned}$$

Compare each of these with some of the Stokes' parameters:

$$s_0 = E_1E_1^* + E_2E_2^*, \quad s_1 = E_1E_1^* - E_2E_2^*, \quad s_2 = 2\,\mathrm{Re}\left(E_1^*E_2\right).$$

We see that we can write the intensities as

$$I_x = \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left(s_0 + s_1\right), \quad I_y = \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left(s_0 - s_1\right), \quad I_/ = \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left(s_0 + s_2\right), \quad I_\backslash = \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}}\left(s_0 - s_2\right).$$

It is then trivial to see that $I_x + I_y = I_/ + I_\backslash$.