Physics 712
Chapter 8 Solutions

3. Show that TEM modes for wave guides with vacuum always have phase velocity $v_p = c$, while TE and TM modes always have phase velocity $v_p > c$. Does this imply you can transmit information faster than light? Perform the appropriate calculation, and show that it never leads to superluminal velocities.

In vacuum, all modes have frequencies given by $\mu_0\varepsilon_0\omega^2 = k^2 + \gamma^2$, where $\gamma = 0$ for TEM modes and $\gamma > 0$ for TE or TM modes. Solving for the angular frequency, we find

$$\omega = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\sqrt{k^2+\gamma^2} = c\sqrt{k^2+\gamma^2}\,.$$

The phase velocity is simply given by

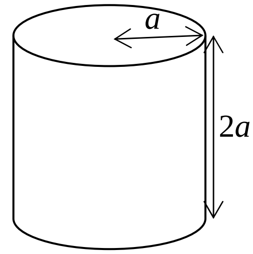
$$v_p = \frac{\omega}{k} = c\frac{\sqrt{k^2+\gamma^2}}{k} = c\sqrt{1+\frac{\gamma^2}{k^2}}\,,$$

So that $v_p = c$ for TEM modes and $v_p > c$ for TE or TM modes. But the phase velocity is *not* the rate at which information is transmitted. This is normally governed by the group velocity, which is given by

$$v_g = \frac{d\omega}{dk} = c\frac{d}{dk}\sqrt{k^2+\gamma^2} = c\frac{k}{\sqrt{k^2+\gamma^2}}\,.$$

This has $v_g = c$ for TEM modes and $v_g < c$ for TE or TM modes. So there is no violation of relativity.

4. Consider a conducting cavity of radius *a* and length 2*a* with nothing (vacuum) inside. Find the energies of the five lowest frequencies ω for this cavity as multiples of *c*/*a*.



The frequencies are simply given by

$$\varepsilon_0\mu_0\omega^2 = \gamma^2 + k^2 = \gamma^2 + \frac{\pi^2 p^2}{(2a)^2},$$

$$\omega = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\sqrt{\gamma^2 + \frac{\pi^2 p^2}{4a^2}} = \frac{c}{a}\sqrt{a^2\gamma^2 + \tfrac{1}{4}\pi^2 p^2}\ .$$

The integer *p* must be positive for TE modes and non-negative for TM modes. The eigenvalues γ take on the values $\gamma = y_{mn}/a$ for TE modes and $\gamma = x_{mn}/a$ for TM modes, where y_{mn} is the *n*'th root of $J'_m(y)$ and x_{mn} is the *n*'th root of $J_m(x)$. The first few values of each of these can be found in the tables above. The resulting frequencies are then

n	$J'_0(y)$	$J'_1(y)$	$J'_2(y)$	$J'_3(y)$	$J'_4(y)$	$J'_5(y)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199

n	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386

$$\omega_{nmp} = \frac{c}{a}\sqrt{x_{nm}^2 + \tfrac{1}{4}\pi^2 p^2}\ \text{for TE modes},\quad \omega_{nmp} = \frac{c}{a}\sqrt{y_{nm}^2 + \tfrac{1}{4}\pi^2 p^2}\ \text{for TM modes}\,.$$

However, recall that *p* must be positive for the TE modes.

Let's compute all the frequencies lower than $4c/a$, which are:

$$\text{TE}_{111}: 2.4202 \quad \text{TE}_{112}: 3.6414 \quad \text{TE}_{211}: 3.4345$$
$$\text{TM}_{010}: 2.4048 \quad \text{TM}_{011}: 2.8724 \quad \text{TM}_{012}: 3.9563 \quad \text{TM}_{110}: 3.8317$$

Listed in order of increasing frequency, we have

$$\text{TM}_{010}: 2.4048 \quad \text{TE}_{111}: 2.4202 \quad \text{TM}_{011}: 2.8724 \quad \text{TE}_{211}: 3.4345 \quad \text{TE}_{112}: 3.6414$$

Those with *m* = 0 are non-degenerate, which in this case, means the first and third one.