Physics 780 – General Relativity

Solutions to Homework D

10. The stress-energy tensor for a perfect fluid is given by $T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}$. All substances we know of have positive energy density, that is, $T^{00} > 0$.

(a) If perfect fluid is going at almost the speed of light, what condition on ρ and/or *p* will assure that $T^{00} > 0$. This is called the *null energy condition*.

The four-velocity is $U^\mu = (\gamma, \gamma\mathbf{v})$, so

$$T^{00} = (\rho + p)U^0 U^0 + p\eta^{00} = \gamma^2(\rho + p) - p = (\gamma^2 - 1)(\rho + p) + \rho .$$

In the limit of high velocity, the $\gamma^2 - 1$ term will be much larger than the other term, so we need $\rho + p > 0$.

(b) What condition on ρ and/or *p* will assure that $T^{00} > 0$ if the fluid is at rest (this is trivial)? Argue that if both the condition from (a) and (b) are true, then $T^{00} > 0$ at *all* speeds. This is called the *weak energy condition*.

If it is at rest, then $T^{00} = \rho$, so we need $\rho > 0$. If we have both conditions, then we see that $T^{00} = (\gamma^2 - 1)(\rho + p) + \rho > 0$.

(c) There is a special relationship between ρ and *p* that makes the stress-energy tensor independent of the fluid's "speed". What is that relationship (it's easier than it sounds). Vacuum energy density acts this way.

To get the velocity-dependance to disappear, we simply make $\rho + p = 0$, or $p = -\rho$.

(d) Radiation has the property that the trace of the stress energy tensor $T = \eta_{\mu\nu}T^{\mu\nu} = 0$. What is the relationship between ρ and/or *p* in this case?

We have

$$T = \eta_{\mu\nu}T^{\mu\nu} = (\rho + p)\eta_{\mu\nu}U^\mu U^\nu + p\eta_{\mu\nu}\eta^{\mu\nu} = -(\rho + p) + 4p = 3p - \rho ,$$

so $p = \frac{1}{3}\rho$.

11. The electromagnetic field produces a stress-energy tensor $T^{\mu\nu} = \varepsilon_0\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right)$**.**

(a) Show that for this stress-energy tensor, the trace is always $T = \eta_{\mu\nu}T^{\mu\nu} = 0$**.**

We have

$$T = \eta_{\mu\nu}T^{\mu\nu} = \varepsilon_0\left(\eta_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda} - \tfrac{1}{4}\eta^{\mu\nu}\eta_{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right) = \varepsilon_0\left(F^{\mu\lambda}F_{\mu\lambda} - \tfrac{1}{4}\cdot 4F^{\lambda\sigma}F_{\lambda\sigma}\right) = 0.$$

(b) Suppose you have a uniform electric field in the *x*-direction, $F^{01} = -F^{10} = E$**. Find all non-vanishing components of the stress-energy tensor in this case.**

We first need to find $F^{\lambda\sigma}F_{\lambda\sigma} = F^{10}F_{10} + F^{01}F_{01} = (-E)E + E(-E) = -2E^2$. Next we note that the second term is non-zero only if we are looking at diagonal terms. The first term must have the three indices μ, ν, and λ all equal to 0 or 1, but λ must not match either μ or ν, which can convince you that we must have $\mu = \nu$. So we have only diagonal components, which are given by

$$T^{00} = \varepsilon_0\left(F^{01}F^{0}{}_{1} - \tfrac{1}{4}\eta^{00}\left(-2E^2\right)\right) = \varepsilon_0\left(E^2 - \tfrac{1}{2}E^2\right) = \tfrac{1}{2}\varepsilon_0 E^2 ,$$

$$T^{11} = \varepsilon_0\left(F^{10}F^{1}{}_{0} - \tfrac{1}{4}\eta^{11}\left(-2E^2\right)\right) = \varepsilon_0\left(-E^2 + \tfrac{1}{2}E^2\right) = -\tfrac{1}{2}\varepsilon_0 E^2 ,$$

$$T^{22} = \varepsilon_0\left(-\tfrac{1}{4}\eta^{22}\left(-2E^2\right)\right) = \tfrac{1}{2}\varepsilon_0 E^2 ,$$

$$T^{33} = \varepsilon_0\left(-\tfrac{1}{4}\eta^{33}\left(-2E^2\right)\right) = \tfrac{1}{2}\varepsilon_0 E^2 ,$$

As a check, we note that $T = \eta_{\mu\nu}T^{\mu\nu} = -T^{00} + T^{11} + T^{22} + T^{33} = 0$, as it must.