Physics 780 – General Relativity

Solution Set E

12. Consider the coordinate transformation in 2D relating Cartesian coordinates to polar,

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}(y/x) \end{aligned}$$

(a) If we have a vector $(V^x, V^y) = (A, 0)$, what is *V* is (V^ρ, V^ϕ)? Write your answers in terms of (ρ, ϕ).

We simply use the coordinate transformation $V'^\mu = \left(\partial x'^\mu / \partial x^\alpha\right) V^\alpha$ to deduce

$$V^\rho = \frac{\partial\rho}{\partial x}V^x + \frac{\partial\rho}{\partial y}V^y = \frac{Ax}{\sqrt{x^2+y^2}} = \frac{A\rho\cos\phi}{\rho} = A\cos\phi,$$

$$V^\phi = \frac{\partial\phi}{\partial x}V^x + \frac{\partial\phi}{\partial y}V^y = \frac{-A\left(y/x^2\right)}{1+y^2/x^2} = \frac{-Ay}{x^2+y^2} = \frac{-A\rho\sin\phi}{\rho^2} = -\frac{A\sin\phi}{\rho}.$$

(b) If we have a 1-form $(V_x, V_y) = (A, 0)$, what is *V* in the new coordinates (V_ρ, V_ϕ)? Write your answers in terms of (ρ, ϕ).

The general transformation is given by $V'_\mu = \left(\partial x^\alpha / \partial x'^\mu\right) V_\alpha$, so we have

$$V'_\rho = \frac{\partial x}{\partial\rho}V_x + \frac{\partial y}{\partial\rho}V_y = A\cos\phi,$$

$$V_\phi = \frac{\partial x}{\partial\phi}V_x + \frac{\partial y}{\partial\phi}V_y = -A\rho\sin\phi.$$

(c) If we have a tensor of type $T^{xx} = A$, with all other components vanishing, what are the components of *T* in the (ρ, ϕ) system? Write your answers in terms of (ρ, ϕ).

We simply follow the same approach, but this time with more factors. We have

$$T^{\rho\rho} = T^{xx}\frac{\partial\rho}{\partial x}\frac{\partial\rho}{\partial x} + \left(T^{xy}+T^{yx}\right)\frac{\partial\rho}{\partial x}\frac{\partial\rho}{\partial y} + T^{yy}\frac{\partial\rho}{\partial y}\frac{\partial\rho}{\partial y} = A\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 = A\left(\frac{\rho\cos\phi}{\rho}\right)^2 = A\cos^2\phi,$$

$$T^{\rho\phi} = T^{\phi\rho} = T^{xx}\frac{\partial\rho}{\partial x}\frac{\partial\phi}{\partial x} = A\frac{x}{\sqrt{x^2+y^2}}\frac{-y/x^2}{1+y^2/x^2} = \frac{-Axy}{\left(x^2+y^2\right)^{3/2}} = \frac{-A\rho^2\cos\phi\sin\phi}{\rho^3} = \frac{-A\cos\phi\sin\phi}{\rho},$$

$$T^{\phi\phi} = T^{xx}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x} = A\left(\frac{-y/x^2}{1+y^2/x^2}\right)^2 = A\left(\frac{y}{x^2+y^2}\right)^2 = \frac{A\rho^2\sin^2\phi}{\rho^4} = \frac{A\sin^2\phi}{\rho^2}.$$

13. Consider the components of a pair of vectors in 2D, $[V,W]^\mu$, where $V = V^\mu\partial_\mu$ and $W = W^\mu\partial_\mu$

(a) Find the commutator if $\left(V^x,V^y\right) = \left(-y,x\right)$ and $\left(W^x,W^y\right) = \left(x,y\right)$. Is it possible that these vectors correspond to partial derivatives of some coordinates (*r*,*θ*), *i.e.*¸is it possible for $V = \partial_\theta$ and $W = \partial_r$ (hint: partial derivatives commute)?

(b) Find the commutator if $\left(V^x,V^y\right) = \left(y,x\right)$ and $\left(W^x,W^y\right) = \left(x,y\right)$. Is it possible that these vectors correspond to partial derivatives of some coordinates (*r*,*θ*)?

Rather than doing these separately, let's say $\left(V^x,V^y\right) = \left(ky,x\right)$ with $k = \mp 1$. We use the formula $[V,W]^\mu = V^\lambda\partial_\lambda W^\mu - W^\lambda\partial_\lambda V^\mu$. We find

$$\begin{aligned}
[V,W]^x &= V^x\partial_x W^x + V^y\partial_y W^x - W^x\partial_x V^x - W^y\partial_y V^x \\
&= ky\partial_x x + x\partial_y x - x\partial_x\left(ky\right) - y\partial_y\left(ky\right) = ky + 0 - 0 - ky = 0, \\
[V,W]^y &= V^x\partial_x W^y + V^y\partial_y W^y - W^x\partial_x V^y - W^y\partial_y V^y \\
&= ky\partial_x y + x\partial_y y - x\partial_x x - y\partial_y x = 0 + x - x - 0 = 0.
\end{aligned}$$

Though this was not what was intended, the conclusion is that in both cases, we find that they commute, and therefore they could be partial derivatives. Though the choices are not unique, it can be shown that we can achieve this in the first case by relating *x* and *y* to the new variables by the relations $\left(x,y\right) = \left(e^\rho\cos\phi, e^\rho\sin\phi\right)$ and in the second case by $\left(x,y\right) = \left(e^\rho\cosh\phi, e^\rho\sinh\phi\right)$. The former coordinates cover the whole range of *x* and *y*, except for the origin. In the latter case, it only covers the region $x > |y|$. The reason it fails at the boundaries at $x = \pm y$ is because on these boundaries, *V* and *W* become parallel or anti-parallel, implying that these directions are not independent, representing a failure of the coordinate system.