Physics 780 – General Relativity

Solution Set L

28. [5] Download the program grcalc from the class website and run it on the metric $ds^2 = h(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$ (or find any other program you can use to calculate it). Compare to the posted answers (or your answers) to problem 25, parts (b) and (c). Do they agree? As your submission, you may send me the Maple worksheet, or any other evidence you did the work.

Only two lines of code need to be changed, namely,

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x := array([r, theta, phi]); N := numelems(x);
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And then

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g := array([[h(r), 0, 0], [0, r^2, 0], [0, 0, r^2*sin(theta)^2]]); ginv := inverse(g);
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It then finds $R_{rr} = \dfrac{h'}{rh}$, $R_{\theta\theta} = \dfrac{rh'}{2h^2} + 1 - \dfrac{1}{h}$, and $R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$, and then $R = \dfrac{2h'}{h^2 r} + \dfrac{2}{r^2} - \dfrac{2}{r^2 h}$, the same as in problem 25.

29. [10] The Einstein equations in 4D, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$ can be rewritten in the form $R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$. Redo this process if we have a cosmological constant, so starting from $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$, rewrite it in the form $R_{\mu\nu} = \cdots$, where there is no curvature on the right side.

We start by taking the trace, which means multiplying by $g^{\mu\nu}$, and use the fact that $g_{\mu\nu}g^{\mu\nu} = 4$. We therefore have

$$R_{\mu\nu}g^{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}g^{\mu\nu} + \Lambda g_{\mu\nu}g^{\mu\nu} = 8\pi G T_{\mu\nu}g^{\mu\nu},$$
$$R - \tfrac{1}{2}\cdot 4R + 4\Lambda = 8\pi G T,$$
$$R = 4\Lambda - 8\pi G T.$$

Substituting this back into the equation we started with, we have

$$R_{\mu\nu} - \tfrac{1}{2}\left(4\Lambda - 8\pi G T\right)g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$
$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu}.$$

30. [15] Assume a static, spherically symmetric spacetime as given below, outside a spherical source (so $T_{\mu\nu}=0$), but include a cosmological constant. The metric is

$$ds^2=-f(r)\mathrm{d}t^2+h(r)\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)$$

(a) Using your formula from problem 29, write down equations for R_{tt}, R_{rr}, and $R_{\theta\theta}$, substituting the explicit forms for $R_{\mu\nu}$ using the handout sssst.pdf

We simply put in the expression from the handout on the left and equate it with $\Lambda g_{\mu\nu}$ on the right, so we have

$$R_{tt}=\frac{f''}{2h}-\frac{f'^2}{4fh}-\frac{f'h'}{4h^2}+\frac{f'}{rh}=-\Lambda f\,,$$

$$R_{rr}=-\frac{f''}{2f}+\frac{f'^2}{4f^2}+\frac{f'h'}{4fh}+\frac{h'}{hr}=\Lambda h\,,$$

$$R_{\theta\theta}=-\frac{f'r}{2fh}+\frac{rh'}{2h^2}+1-\frac{1}{h}=\Lambda r^2\,,$$

$$R_{\phi\phi}=R_{\theta\theta}\sin^2\theta=\Lambda r^2\sin^2\theta\,.$$

The last two equations are identical, of course, so we don't need to work with them separately.

(b) Combining $hR_{tt}+fR_{rr}$, show that in this case you *still* can show that fh is a constant, which can be chosen to be 1.

Combining them as suggested, we note that the terms on the right side cancel, so we get the same equations as before:

$$hR_{tt}+fR_{rr}=\frac{f''}{2}-\frac{f'^2}{4f}-\frac{f'h'}{4h}+\frac{f'}{r}-\frac{f''}{2}+\frac{f'^2}{4f}+\frac{f'h'}{4h}+\frac{fh'}{hr}=-\Lambda hf+\Lambda hf\,,$$

$$\frac{f'h+fh'}{hr}=0\,.$$

The last expression tells us that $(fh)'=0$, so fh is constant. We can then rescale our time coordinate to make this product come out to one, so $fh=1$.

(c) Rewrite the $R_{\theta\theta}$ equation by replacing $h=f^{-1}$.

Since $f=1/h$, it is easy to see that $f'=-h'/h^2$, so we can use this to rewrite this equation as

$$R_{\theta\theta}=-\frac{f'r}{2}-\frac{f'r}{2}+1-f=-f'r+1-f=\Lambda r^2\,.$$

(d) Solve the $R_{\theta\theta}$ equation to find *f*(*r*).

We rearrange slightly and find

$$0 = -f'r + 1 - f - \Lambda r^2 = \frac{d}{dr}\left(r - fr - \tfrac{1}{3}\Lambda r^3\right).$$

Since the derivative of this expression is zero, its value is a constant we name R_S. Then we solve for f:

$$\begin{aligned} r - fr - \tfrac{1}{3}\Lambda r^3 &= R_S\,, \\ fr &= r - R_s - \tfrac{1}{3}\Lambda r^3\,, \\ f(r) &= 1 - \frac{R_S}{r} - \frac{1}{3}\Lambda r^2\,. \end{aligned}$$

The parameter R_S would normally be written as $2GM$, so then our final metric is

$$ds^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2\right), \quad \text{where} \quad f(r) = 1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2\,.$$