Physics 780 – General Relativity

Solution Set M

Assume you are working in the Schwarzschild metric for all of these problems.

31. [5] Consider a massive particle moving in this metric.

(a) [3] Show that if at any time $U^\theta = U^\phi = 0$, then from the geodesic equations we will also have $dU^\theta / d\tau = dU^\phi / d\tau = 0$, so this will continue to be true indefinitely.

We use the geodesic equation, and use the values of the Christoffel symbols as provided in the handout:

$$\frac{d}{d\tau}U^\theta = -\Gamma^\theta_{\mu\nu}U^\mu U^\nu \quad \text{and} \quad \frac{d}{d\tau}U^\phi = -\Gamma^\phi_{\mu\nu}U^\mu U^\nu .$$

The only nonzero components of U are U^r and U^t. But the non-vanishing Christoffel symbols have an even number of ϕ's, so the second expression vanishes. They also have even an even number of θ's, or one θ and two ϕ's, so the first one also vanishes, so $dU^\theta / d\tau = dU^\phi / d\tau = 0$.

(b) [2] Show that if $\theta = \frac{1}{2}\pi$ and $U^\theta = 0$, then from the geodesic equations we will also have $dU^\theta / d\tau = 0$, so this will continue to be true indefinitely.

By the same argument, the only non-vanishing term will come from $-\Gamma^\theta_{\phi\phi}U^\phi U^\phi$, but $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, which vanishes at $\theta = \frac{1}{2}\pi$.

32. [15] Consider a massive particle (so $U_\mu U^\mu = -1$) starting at rest near $r = \infty$, so $U^\theta = U^\phi = U^r = 0$.

(a) [2] If $U^r = 0$ at infinity, what is the value of U^t and of $U_t = -E$? Recall that E is a constant.

We have

$$-1 = U_\mu U^\mu = -f(r)\left(U^t\right)^2 = -\left(1 - 2GM/r\right)\left(U^t\right)^2 .$$

But we are at $r = \infty$, so this gives us $U^t = 1$ and therefore $U_t = -E = -1$. So this corresponds to to $E = 1$, which will remain through this problem.

(b) [4] Find a formula for U^t and for U^r as a function of *r*.

We are no longer at infinite radius, so now $U^t = g^{tt}U_t = -f(r)^{-1}E = -f(r)^{-1}$. We can then get the radial velocity from $U_\mu U^\mu = -1$, namely

$$-1 = U_\mu U^\mu = U_t U^t + U_r U^r = -\frac{1}{f(r)} + h(r)\left(U^r\right)^2 = \frac{1}{f(r)}\left[\left(U^r\right)^2 - 1\right],$$

$$\left(U^r\right)^2 = -f(r) + 1 = \frac{2GM}{r}.$$

$$U^r = -\sqrt{\frac{2GM}{r}} \quad \text{and} \quad U^t = \frac{1}{1 - 2GM/r}.$$

We chose the minus sign on the velocity, since it is falling inwards.

(c) [4] Take your formula from part (b) for $U^r = dr/d\tau$ and integrate it over radius to find out how much proper time it takes to fall from a distance *r* down to *r* = 0. Assume the formulas work right through *r* = 2*GM*, despite the apparent singularity of the metric there.

The time required to go from *r* to 0 is given by

$$\tau = \int_r^0 \frac{d\tau}{dr}dr = -\int_0^r \left(-\sqrt{\frac{r}{2GM}}\right)dr = \frac{2r^{3/2}}{3\sqrt{2GM}}.$$

(d) [4] How long does it take after you cross the Schwarzschild radius *r* = 2*GM* to reach the origin for the black hole at the center of our galaxy, with a mass of 4.4×10^6 solar masses. The Schwarzschild radius for the Sun was found in problem set A, and is 2.95 km.

Substituting *r* = 2*GM*, we find $\tau = \frac{2}{3}(2GM)$. Substituting the explicit mass, we have

$$\tau = \frac{2}{3}(2GM) = \frac{2}{3}\left(4.4\times10^6\right)\left(2GM_\odot\right) = \frac{2}{3}\left(4.4\times10^6\right)\left(2950\text{ m}\right) = \frac{8.65\times10^9\text{ m}}{2.998\times10^8\text{ m/s}} = 28.9\text{ s}.$$

I hope you enjoy your last half-minute of life, though you in fact will be spaghettified before you reach the singularity.

33. [10] Consider a massive particle of arbitrary energy moving in the $\theta = \frac{1}{2}\pi$ plane.

(a) [7] A circular orbit is possible whenever there is a local maximum or local minimum of the effective potential. Find a formula for the two radii where circular orbits are possible in terms of *J* and *M*. Which of these is stable, and which unstable?

We start with our equation for orbits, namely

$$\frac{1}{2}\left(E^2 - 1\right) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}\left(r\right), \quad \text{where} \quad V_{eff}\left(r\right) = -\frac{GM}{r} + \frac{J^2}{2r^2} - \frac{GMJ^2}{r^3}$$

A circular orbit is any situation where V_{eff} is flat and $\frac{1}{2}\left(E^2 - 1\right) = V_{eff}\left(r\right)$. To be flat, we take the first derivative of our potential and set it to zero, so

$$0 = \frac{GM}{r^2} - \frac{J^2}{r^3} + \frac{3GMJ^2}{r^4},$$
$$0 = GMr^2 - J^2 r + 3GMJ^2 .$$

We solve this with the quadratic formula and find

$$r_\pm = \frac{J^2 \pm \sqrt{J^4 - 12G^2M^2J^2}}{2GM} = \frac{J^2 \pm J\sqrt{J^2 - 12G^2M^2}}{2GM}.$$

You can figure out which one is stable by looking at the potential; a minimum is stable but a maximum is unstable. You can simply look at a graph, such as appears in slide 18 of chapter 5, or you can reason it out by looking at the potential. The potential is zero at infinity, but then it decreases due to the first term (which dominates at large *r*). This term will (if *J* is large enough) eventually be overcome by the second term, so the outermost (or most positive) value of *r* will be a minimum (stable). But eventually the final term in the potential will take over, making it go back towards negative numbers, so the inner solution will be a local maximum. So r_+ is stable and r_- is unstable.

(b) [3] As you decrease *J*, the two radii found in part (a) move together and merge. What is the value of *J*, and the corresponding value of *r* when this happens, in terms of *M*? This is called the innermost stable circular orbit, or ISCO for this metric.

When $J^2 = 12G^2M^2$, or $J = 2\sqrt{3}GM$ we have $r_+ = r_-$. You can show that in this case the second derivative vanishes, as the two merge at

$$r_\pm = \frac{12G^2M^2}{2GM} = 6GM .$$

Though this is called the innermost stable circular orbit, in fact it is *unstable* because if you are exactly in this orbit, the third derivative of the potential is *not* zero, so any small perturbation will cause the object to plummet into the black hole.