Physics 780 – General Relativity

Solution Set T

47. The universe will be finite in size if Ω > 1.

(a) The value given in class for the density parameter is $\Omega_0 = 0.9993 \pm 0.0037$. Taking this literally, for a closed universe, what is the *smallest* possible value for $H_0 a_0$ assuming $1 < \Omega \le 0.9993 + 0.0037$?

The first Friedmann equation says that

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}.$$

The left side evaluated today is Hubble's constant, and the ratio of the first term on the right to the expression on the left is Ω, so evaluating everything today, we have

$$H_0^2 = H_0^2 \Omega_0 - \frac{k}{a_0^2},$$

$$\frac{k}{a_0^2} = H_0^2 \left(\Omega_0 - 1\right).$$

We are interested in the case where the universe is closed, $\Omega_0 > 1$, which corresponds to $k = +1$. We see that $\Omega_0 \le 1.0030$ therefore leads to

$$\frac{1}{a_0^2} \le H_0^2 \left(1.0030 - 1\right),$$

$$a_0^2 H_0^2 \ge \frac{1}{0.0030} = 333,$$

$$a_0 H_0 \ge \sqrt{333} = 18.2.$$

Keeping in mind that we know H_0 actually has units of s^{-1}, and a_0 has units of m, the left hand side has units of velocity. Since we are allowed to multiply or divide by c as necessary, the correct inequality must be $a_0 H_0 \ge 18.2c$.

(b) What is the spatial volume for a closed universe with scale factor *a*? You will probably have to use the version of the metric in terms of *ψ* to get the full range $\psi \in [0, \pi]$ of the whole universe.

The metric when written in terms of ψ, where $r = \sin\psi$, is

$$ds^2 = -\mathrm{d}t^2 + a^2\left[\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right].$$

The metric is diagonal, and the space part of the metric is $g_{\psi\psi} = a^2$, $g_{\theta\theta} = a^2 \sin^2\psi$ and $g_{\phi\phi} = a^2 \sin^2\psi \sin^2\theta$. The volume of the universe is just the integral of $\sqrt{g^{(3)}}$, where $g^{(3)}$ is the determinant of the space part of the metric, so we have

$$V = \int \sqrt{a^2 a^2 \sin^2\psi a^2 \sin^2\psi \sin^2\theta}\, d\psi\, d\theta\, d\phi = a^3 \int_0^\pi \sin^2\psi\, d\psi \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$
$$= a^3 \left(\tfrac{1}{2}\pi\right)(2)(2\pi) = 2\pi^2 a^3 .$$

(c) Write the scale factor *a* from part (a) in Gpc if $H_0 = 67.7\ \text{km/s/Mpc}$. Don't forget to add factors of *c* to get the units right! Then find the minimum volume of the visible universe in Gpc³ using the result of part (b).

We already put *c* in as appropriate, so we simply start computing:

$$a_0 \geq \frac{18.2c}{H_0} = \frac{18.2\left(2.998\times 10^5\ \text{km/s}\right)}{67.7\ \text{km/s/Mpc}} = 80{,}810\ \text{Mpc} = 80.81\,\text{Gpc}.$$

We now just put it in the formula from part (b) to find

$$V_0 = 2\pi^2 a_0^3 \geq 2\pi^2 \left(80.81\ \text{Gpc}\right)^3 = 1.042\times 10^7\ \text{Gpc}^3 .$$

48. We found an integral formula for the current age of the universe times the current Hubble constant t_0H_0 and the current density parameter if there is *only* matter with density Ω_m.

(a) Repeat this exercise and find $t_0 H_0$ if there is *only* radiation with density Ω_r. Perform the integral.

The current density of radiation is given by $\frac{8}{3}\pi G\rho_{r0} = H_0^2\Omega_r$. Because the density of radiation falls as a^{-4}, the density at other times must be

$$\frac{8\pi}{3} G\rho_r = \frac{8\pi}{3} G\rho_{r0}\left(\frac{a_0}{a}\right)^4 = H_0^2\Omega_r\left(\frac{a_0}{a}\right)^4$$

We will use the first Friedmann equation, which tells us that

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho_r - \frac{k}{a^2}.$$

Evaluating this equation today, we would have

$$H_0^2 = H_0^2\Omega - \frac{k}{a_0^2},$$

$$\frac{k}{a_0^2} = H_0^2\left(\Omega_r - 1\right)$$

Substituting this back into the first Friedmann equation, we have

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho_r - \frac{k}{a^2} = H_0^2 \Omega_r \frac{a_0^4}{a^4} - H_0^2 \left(\Omega_r - 1\right) \frac{a_0^2}{a^2}.$$

Defining $x = a/a_0$, this equation becomes

$$\frac{\dot{x}^2}{x^2} = H_0^2 \left(\frac{\Omega_r}{x^4} + \frac{1-\Omega_r}{x^2} \right).$$

We solving for dx/dt, we find

$$\frac{dx}{dt} = H_0 \sqrt{\Omega_r / x^2 + 1 - \Omega_r}.$$

We want the time, which we will get by rearranging and then integrating. Note that since $x = a/a_0$, we want to integrate a from when the universe was size zero, $x = 0$, to now $x = 1$.

$$H_0 dt = \frac{dx}{\sqrt{\Omega_r / x^2 + 1 - \Omega_r}},$$

$$H_0 t_0 = \int_0^1 \frac{dx}{\sqrt{\Omega_r / x^2 + 1 - \Omega_r}}.$$

We now have to integrate this, which isn't so bad:

$$H_0 t_0 = \int_0^1 \frac{x\,dx}{\sqrt{\Omega_r + \left(1-\Omega_r\right) x^2}} = \frac{1}{2\left(1-\Omega_r\right)} \int_0^1 \frac{d\left[\left(1-\Omega_r\right) x^2\right]}{\sqrt{\Omega_r + \left(1-\Omega_r\right) x^2}} = \left. \frac{\sqrt{\Omega_r + \left(1-\Omega_r\right) x^2}}{1-\Omega_r} \right|_0^1$$

$$= \frac{1-\sqrt{\Omega_r}}{1-\Omega_r} = \frac{1-\sqrt{\Omega_r}}{\left(1-\sqrt{\Omega_r}\right)\left(1+\sqrt{\Omega_r}\right)} = \frac{1}{1+\sqrt{\Omega_r}}.$$

Technically, this integration was done under the assumption that $\Omega_r \neq 1$, but it is trivial to redo the integral in this case and verify that the final result is still valid.

(b) Repeat this exercise and find $t_0 H_0$ if there is radiation with density Ω_r and matter with density Ω_m, but the universe is flat, so $\Omega_r + \Omega_m = 1$. Perform the integral. If needed, you can use the fact that $\Omega_r + \Omega_m = 1$ and the fact that they are both positive to conclude they are both smaller than 1.

Since the universe is flat, $k = 0$, so the first Friedmann equation gives

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho_r + \frac{8\pi}{3} G\rho_m.$$

The current value of each of the terms on the right are $\frac{8}{3}\pi G\rho_r = H_0^2 \Omega_r$ and $\frac{8}{3}\pi G\rho_m = H_0^2 \Omega_m$. Using the fact that $\rho_r \propto a^{-4}$ and $\rho_m \propto a^{-3}$, the Friedmann equation yields

$$\frac{\dot{a}^2}{a^2} = H_0^2\Omega_r \frac{a_0^4}{a^4} + H_0^2\Omega_m \frac{a_0^3}{a^3}.$$

As usual, let $x = a/a_0$, then this equation becomes

$$\frac{\dot{x}^2}{x^2} = \frac{H_0^2\Omega_r}{x^4} + \frac{H_0^2\Omega_m}{x^3},$$

$$\frac{dx}{dt} = H_0\sqrt{\frac{\Omega_r}{x^2} + \frac{\Omega_m}{x}}.$$

We rearrange this and integrate it over x to find t_0:

$$H_0 dt = \frac{dx}{\sqrt{\Omega_r/x^2 + \Omega_m/x}},$$

$$H_0 t_0 = \int_0^1 \frac{dx}{\sqrt{\Omega_r/x^2 + \Omega_m/x}} = \int_0^1 \frac{xdx}{\sqrt{\Omega_r + \Omega_m x}} = \frac{1}{\Omega_m^2}\int_0^1 \frac{(\Omega_m x + \Omega_r - \Omega_r)\,d(\Omega_m x + \Omega_r)}{\sqrt{\Omega_r + \Omega_m x}}$$

$$= \frac{1}{\Omega_m^2}\int_0^1 \left(\sqrt{\Omega_m x + \Omega_r} - \frac{\Omega_r}{\sqrt{\Omega_m x + \Omega_r}}\right) d(\Omega_m x + \Omega_r)$$

$$= \frac{1}{\Omega_m^2}\left[\tfrac{2}{3}(\Omega_m x + \Omega_r)^{3/2} - 2\Omega_r\sqrt{\Omega_m x + \Omega_r}\right]_0^1$$

$$= \frac{1}{\Omega_m^2}\left[\tfrac{2}{3} - 2\Omega_r - \tfrac{2}{3}\Omega_r^{3/2} + 2\Omega_r^{3/2}\right] = \frac{2}{3\Omega_m^2}\left(1 - 3\Omega_r + 2\Omega_r^{3/2}\right),$$

we used $\Omega_r + \Omega_m = 1$ when substituting in the upper limits.

The final expression can be simplified, somewhat, by factoring the numerator as a cubic in $\sqrt{\Omega_r}$, and in the denominator rewriting $\Omega_m = 1 - \Omega_r = \left(1 - \sqrt{\Omega_r}\right)\left(1 + \sqrt{\Omega_r}\right)$, so we have

$$H_0 t_0 = \frac{2\left(1 + 2\sqrt{\Omega_r}\right)\left(1 - \sqrt{\Omega_r}\right)^2}{3\left(1 - \sqrt{\Omega_r}\right)^2\left(1 + \sqrt{\Omega_r}\right)^2} = \frac{2\left(1 + 2\sqrt{\Omega_r}\right)}{3\left(1 + \sqrt{\Omega_r}\right)^2}.$$

49. Way back in the previous millennium (i.e., pre-1995) we only knew about matter (and a tiny bit of radiation), and were none too confident about the value of Ω_m . For *this* problem, assume the universe contains matter only.
    (a) Show that if $\Omega_m \le 1$, the universe will never stop growing, *i.e.*, there is no point in the future when $\dot{a} = 0$.
    (b) Show that if $\Omega_m > 1$, it is inevitable that the universe will eventually stop growing. Find a formula for the size of the universe compared to now, a/a_0 , when the universe will stop growing as a function of Ω_m .