Physics 745 - Group Theory

Homework Set 23

Due Wednesday, April 1

1. The group $SU(2)$ shows up in surprising places. Consider, for example, the two-dimensional harmonic oscillators, which can be written in the form

$$H = \hbar\omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)$$

where a_1 and a_2 are two operators satisfying

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \quad \left[a_i, a_j\right] = \left[a_i^\dagger, a_j^\dagger\right] = 0$$

By conventional means, it is not hard to show that this Hamiltonian results in degenerate eigenvalues. But why? Is there a symmetry which results in this degeneracy?

(a) Define the three operators

$$\mathcal{T}_1 = \tfrac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \quad \mathcal{T}_2 = \tfrac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right), \quad \mathcal{T}_3 = \tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right).$$

Show that these operators satisfy the $SU(2)$ commutation relations,

$$\left[\mathcal{T}_a, \mathcal{T}_b\right] = i\sum_c \varepsilon_{abc}\mathcal{T}_c \ .$$

This is three relations in all.

(b) Show that all three of the operators commute with the Hamiltonian

(c) There is a simple relationship between the Hamiltonian and the generators, namely

$$\mathcal{T}^2 = \mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2 = \tfrac{1}{4}\left[H^2/\hbar^2\omega^2 - 1\right]$$

Demonstrating this is straightforward but laborious. Using this relationship, find the possible eigenvalues of H, and their degeneracy, using only your knowledge of the eigenvalues of $\mathcal{T}^2$.